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STRUCTURAL DYNAMICS OF HELICOPTOR ROTOR HAVING PRECONE-PRESWEEP-PREDROOP-PRETWIST AND TORQUE OFFSET INCLUDING HUB MOTIONS

by

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CERTIFICATE

Certified that the work contained in the thesis entitled *"STRUCTURAL DYNAMICS OF HELICOPTOR ROTOR HAVING PRECONE-PRESWEEP-PREDROOP-PRETWIST AND TORQUE OFFSET INCLUDING HUB MOTIONS."*, by "*Punit Kumar Gupta*", has been carried out under my supervision and that this work has not been submitted elsewhere for a degree.

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DEDICATED

TO

MY PARENTS

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ABSTRACT

The purpose of the present study is to develop a most general structural dynamic model for a flexible rotor blade undergoing coupled axial, flap, lag and torsional deformations. The dynamic model includes the complex geometric parameters, such as pretwist, predroop, precone, presweep and torque offset, as well as hub motion. The equations of motions are derived using the Hamilton's principle. Using a beam type finite element model for the blade, the mass and stiffness matrices are obtained respectively, from the kinetic energy and the strain energy expressions.
The structural dynamic model derived in this study was then used to calculate the undamped natural frequencies of a rotating blade. These results are compared with the data available in the literature.

This dynamic model will be useful for the study of active control of vibration in helicopters as well as for rotor blade aeroelastic analysis.

NOMENCLATURE

a	Torque offset
e_1, e_2	Root offset
$\hat{e}_x, \hat{e}_y, \hat{e}_z$	Unit vector along X, Y and Z axes
E_o	Reference modulus
$Im_{\eta\eta}, Im_{\zeta\zeta}, Im_{\eta\zeta}$	Mass moments of inertia of the beam cross-section
$[K^{cf}]$	Centrifugal stiffening matrix
$[K^E]$	Linear stiffness matrix
l	Length of the blade
l_e	Length of each finite element ($= \frac{l}{N}$)
m	Mass per unit length of the blade
$m\eta_m, m\zeta_m$	First moments of mass of the beam cross-section
$[M]$	Mass matrix
$[M^1], [M^2],$	Matrices defined for writing the variation of kinetic
$[M^3], [M^4]$	energy in matrix form
$[M^C]$	Coriolis damping matrix
N_b	Number of blades in the rotor system
N	Number of finite elements
O_H	Hub centre
{q}	Vector of finite element nodal displacements
$\vec{r}_p$	Position vector of a point 'p' on the deformed k^{th} blade
R_x, R_y, R_z	Components of perturbational hub motion

t	Time
T	Kinetic energy
$[T_{ij}]$	Transformation matrix between orthogonal co-ordinate systems i and j
u_k, v_k, w_k	k^{th} blade deformation in axial, lead-lag and flap directions
U	Strain energy
V_H	Hub velocity
$\vec{V}$	Velocity of a point 'p' on k^{th} blade
V_x, V_y, V_z	Component of $\vec{V}$ in X, Y and Z direction
$\{V^L\}, \{V^{NL}\}, \{V^I\}$	Vectors defined in the expression of kinetic energy variation
$\{V\}, \{W\}, \{\Phi\}, \{U\}$	Vectors of element nodal degrees of freedom
v'_k	$= \frac{d(v_k/l)}{d(x/l)}$
w'_k	$= \frac{d(w_k/l)}{d(x/l)}$
W_e	External work due to nonconservative forces
x_k	Coordinate along k^{th} blade axis
x, y, z	Coordinate of a point in the $\hat{e}_x - \hat{e}_y - \hat{e}_z$ system
x, η, ζ	Coordinate of a point in $\hat{e}_x - \hat{e}_\eta - \hat{e}_\zeta$ system
$\bar{x}$	$= \frac{x}{l}$
$Z_u, Z_v, Z_w,$ $Z_\phi, Z'_v, Z'_w,$ $\bar{Z}_u, \bar{Z}_v, \bar{Z}_w,$ $\bar{Z}_u, \bar{Z}'_v, \bar{Z}'_w$	Notation used for writing the beam kinetic energy in concise form
β_d	Blade predroop angle
β_k	Local slope in flap bending of k^{th} blade
β_p	Blade precone angle
β_s	Blade presweep angle
ξ_k	Local slope in lag bending of k^{th} blade
ϵ	Non dimensional parameter representing the order of magnitude of typical elastic blade bending slope
$\epsilon_{xx}, \epsilon_{x\eta}, \epsilon_{x\zeta}$	Strain components

$\sigma_{xx}, \sigma_{x\eta}, \sigma_{x\zeta}$	Stress components
α	Warping amplitude
τ_o	Initial twist rate of the blade
θ_G, θ_g	Geometric pitch in k^{th} blade
ϕ_k	Elastic twist
θ_I	Control pitch input
$\theta_x, \theta_y, \theta_z$	Rigid body perturbational rotation in roll-pitch-yaw
ρ	Density
$\{\Phi_c\}, \{\Phi_q\}$	Arrays of Hermite cubic and quardratic interpolation polynomials respectively
$\{\Phi_c'\}, \{\Phi_q'\}$	First derivative of $\{\phi_c\}$ and $\{\phi_q\}$ with respect to x
ψ_k	Azimuthal angle of k^{th} blade
ψ	Non-dimensional time ($\psi = \Omega\, t$)
Ψ	Cross-sectional warping function
$\vec{\omega}_k$	Angular velocity of k^{th} blade
Ω	Speed of rotation of rotor
$\omega_x, \omega_y, \omega_z$	Components of $\vec{\omega}_k$ in x, y and z direction
$(\ )'$	Differentiation of ( ) with respect to x
d( )	Differential of ( )
$(\ )_\eta, (\ )_\zeta$	Differentiation with respect to η and ζ
$(\ )_x, (\ )_{xx}$	Differentiation with respect to x of variables u, v, w and ϕ
$\delta(\ )$	Variation of ( )
$(\dot{\ }), (\ddot{\ })$	$\frac{\partial}{\partial\psi}, \frac{\partial}{\partial\psi^2}$
{ }	Vector
[ ]	Matrix
$1, 2, 3, 4, 5, 6$	Quantities refer to the corresponding coordinate system

Chapter 1

INTRODUCTION

The reduction of vibration levels in helicopters is one of the key problems in the design of helicopter. Increasing demand towards expanding the mission range of helicopter, such as high speed, high 'g' manoeuvres , coupled with improving the system reliability and reduced maintenence cost, has resulted in more stringent vibration specifications. Hence, the reduction of vibratory levels below an acceptable limit has become a critical design consideration in modern helicopters. Over the years, vibratory levels in the helicopter cabin have been reduced by suitable design modifications and also by incorporating vibration absorbers and isolators. However, in the next generation helicopters, with the stringent vibration control, it will become necessary to reduce the vibration levels below 0.05 g or even 0.02 g.

It is well known that the time varying loads on the main rotor system contributes significantly to the vibration in helicopter. Therefore, the structural dynamic characteristics of the rotor blade and also the dynamic characteristics of the fuselage have a very strong influence on the vibratory levels in helicopters. Any analytical study of vibration in the helicopters requires the developement of a dynamic model of the helicopter system. The major components of this model are:

- Rotor blade model
- Fuselage model

- Rotor-Fuselage interface model

The formulation of the *rotor blade model* requires the development of structural, inertial and aerodynamic operators associated with the rotor blade motion. The *helicoptor fuselage model* is represented by an idealized structural dynamic model of a three-dimensional structure. The *rotor-fuselage interface model* must represent both the geometry of the interface as well as the aerodynamic interaction in an appropriate manner.

This study deals with the formulation and solution of a structural dynamic (structural and inertia operators) model of a rotor blade incorporating all complex geometric parameters of a realistic rotor system.

1.1 STRUCTURAL MODELLING OF ROTOR BLADE

Most of the published work on the structural modelling of the helicopter rotor blades assume a beam type model. Nonlinear beam kinematics, which incorporate small strains and finite (moderate or large) rotations, are being used to account for the coupling effect between axial, bending and torsional deformations. In the derivation of the strain-displacement relationship, a small strain assumption (i.e. strains are small as compared to unity) is made. Such an assumption is consistent with the design requirement based on fatigue life consideration which states that the rotor blades must be designed to have an operating strain level well below the elastic limit of the blade material.

The first analytical model for the flap-lag-torsion of pretwisted nonuniform rotor blades was developed by Houbolt and Brooks[1]. This model is based on a linear theory which neglects all the nonlinear displacement terms in the derivation. As a result, the bending-torsion coupling effects due to the geometric nonlinearities,

which are important for the rotor blade analysis, are absent in the model.
In order to incorporate the geometric nonlinearities due to the assumption of small strains and finite rotations, one should distinguish between the deformed and undeformed configurations of the blade, and derive the transformation relation between the triad of unit vectors associated with the undeformed configuration of the blade and the triad of unit vectors associated with the deformed blade configuration. In the moderate deflection beam theories [2-7], the transformation relations between deformed and the undeformed triad of unit vector were derived. These beam theories were later used to formulate the structural, inertial and aerodynamic operators for the aeroelastic stability analysis of rotor blades [8-12]. Structural modelling for composite rotor blade has also been developed recently by various investigators[13-22]. The earlier models presented in Refs. 2-7, were restricted to treatment of isotropic blades. In general, these models did not include the effects of cross-sectional transverse shear and warping. On the other hand, the composite blade models include these effects.
During the derivation of the moderate deflection beam theory, a number of nonlinear terms are generated. Many of them are relatively small due to the assumption of the small deflections and moderate rotation. Therefore, an ordering scheme is needed to identify and neglect higher order nonlinear terms in a consistent manner. The commonly used ordering scheme [23,24] is based on assigning orders of magnitude to the nondimensional physical parameters governing the aeroelastic problem in terms of blade bending slopes, which are assumed to be of the order of ϵ. A second order approximation assumes that the terms of order ϵ^2 are neglected compared to the terms of order 1. In this study a third order approximation has been used, i.e, terms of order ϵ^3 are neglected compared to the terms of order 1. i.e.

$$1 + O(\epsilon^3) \approx 1$$

In all the studies reported in Refs.[2-25], only certain geometric parameters such as precone, hinge offset and tip sweep have been included in modelling the rotor blade dynamics. In addition, most of these models, except [24,25], do not consider

hub motion. In the present formulation, an attempt has been made to include all the complex geometric parameters such as torque offset, blade root offset, precone angle, predroop angle, presweep angle and pretwist, as well as hub motion.

1.2 OBJECTIVES

The objectives of the present study are:

- To develop a most general model for the rotor blade including various geometric parameters like root offset, torque offset, precone, predroop, presweep, and pretwist and hub motions.

- To validate the model by comparing the results of this study with those available in the litrature.

Chapter 2

MODEL DESCRIPTION AND ORDERING SCHEME

Helicopter rotor blades are long slender beams attached to the hub through a mechanical arrangement. There are several geometrical parameters describing the configuration of the rotor blade attachment to the hub. A schematic of the rotor configuration is shown in Fig.1. Figure 1 shows the torque offset 'a' which is the distance from the centre of rotation (hub centre) to the pitch axis of the blade. 'e_1' and 'e_2' refer to the blade root offset distance from the centre of the hub. β_p represents the precone angle defining the orientation of the blade pitch control axis with respect to the hub plane of rotation. β_s is the presweep angle defining the orientation of the blade in the plane of rotation. β_d represents the predroop angle representing the inclination of the blade with respect to the pitch control axis of the blade.

2.1 BASIC ASSUMPTIONS

In the formulation of the dynamic model of the rotor blade, several assumptions have been made which are given below:

1. The blade is treated as an elastic beam.

2. The blade is modeled by a series of straight beam finite elements located along the elastic axis of the blade.

3. The speed of rotation (Ω) of rotor is constant.

4. The rotor shaft is rigid.

5. The blade undergoes moderate deformation.

6. The blade cross section can have a general shape with distinct shear centre, torque centre and the centre of mass.

2.2 ORDERING SCHEME

In the formulation of the equations of motion of a rotor blade undergoing moderate deformations, a large number of higher order terms are generated. In order to identify and eliminate higher order terms in a consistent manner, an ordering scheme is employed. This ordering scheme is based on the assumption that the slopes of the deformed elastic blade are moderate, and of order ϵ $(0.10 \leq \epsilon \leq 0.20)$. Orders of magnitude are then assigned to various non-dimensional physical parameters governing the rotor blade dynamic problem, in terms of ϵ. In the derivation of the governing equations, it is assumed that terms of the order ϵ^3 are neglected with respect to terms of order 1, i.e.,

$$O(1) + O(\epsilon^3) \approx O(1)$$

The order of magnitude of various non-dimensional parameters governing this problem are given below :

<u>ORDER</u> 1

$\cos \phi_k, \sin \phi_k, \frac{x_k}{l} = O(1)$

$\frac{1}{\Omega}\frac{\partial}{\partial t}(\ ) = \frac{\partial}{\partial \phi}(\ ) = O(1)$

$l\frac{\partial}{\partial x_k}(\ ) = \frac{\partial}{\partial \bar{x}_k} = O(1)$

<u>ORDER</u> $\epsilon^{1\backslash 2}$

$\theta_{GK} = O(\epsilon^{1\backslash 2})$

<u>ORDER</u> ϵ

$\frac{a}{l}, \frac{e_1}{l}, \frac{e_2}{l}, \frac{\eta}{l}, \frac{\zeta}{l}, \frac{v_k}{l}, \frac{w_k}{l} = O(\epsilon)$

$v_k', w_k', \phi, \beta_p, \beta_d, \beta_s = O(\epsilon)$

<u>ORDER</u> $\epsilon^{3\backslash 2}$

$\frac{R_x}{l}, \frac{R_y}{l}, \frac{R_z}{l}, \theta_x, \theta_y, \theta_z = O(\epsilon^{3\backslash 2})$

$Im_{\eta\eta}, Im_{\zeta\zeta} = O(\epsilon^{3\backslash 2})$

<u>ORDER</u> ϵ^2

$\frac{u_k}{l}, u_k', m\eta_m, m\zeta_m = O(\epsilon^2)$

It is important to note that, ordering schemes are based on physical understanding of the behaviour of actual blade configurations. Hence care must be excercised in deleting the higher order terms, based on this ordering scheme.

Chapter 3

COORDINATE SYSTEMS

The description of the complex deformation of a rotor blade requires several coordinate systems. The transformation relation between quantities referred in various inertial, non-inertial coordinate systems is to be established before deriving the equations of motion of the rotor blade. The relation between two orthogonal systems with axes X_i, Y_i, Z_i and X_j, Y_j, Z_j with $\hat{e}_{xi}, \hat{e}_{yi}, \hat{e}_{zi}$ and $\hat{e}_{xj}, \hat{e}_{yj}, \hat{e}_{zj}$ as unit vectors along the respective axes can be written as

$$\left\{ \begin{array}{c} \hat{e}_{xi} \\ \hat{e}_{yi} \\ \hat{e}_{zi} \end{array} \right\} = [T_{ij}] \left\{ \begin{array}{c} \hat{e}_{xj} \\ \hat{e}_{yj} \\ \hat{e}_{zj} \end{array} \right\} \tag{3.1}$$

where the transformation matrix $[T_{ij}]$ can be obtained using the Euler angles required to rotate the j-system so as to make it parallel to i-system. The coordinate systems used in deriving the equation of motion for the rotor model are described below:

3.1 INERTIAL SYSTEM -R

The coordinate system 'R', shown in Fig.2, has its origin at the centre O_H of the unperturbed hub. The x_R axis is pointing towards the helicoptor tail and z_R is

pointing upwards. The unit vectors along the three axes are $\hat{e}_{xR}$, $\hat{e}_{yR}$, $\hat{e}_{zR}$.

3.2 BODY FIXED HUB SYSTEM -H

The coordinate system H shown in Fig.3, is a body fixed system with its origin fixed at the centre of rotor hub O_H. Prior to perturbational motion of the hub, the H -system coincides with R -system.
If θ_x, θ_y and θ_z represent the roll-pitch-yaw rotations, then the transformation matrix $[T_{HR}]$ can be written as -

$$[T_{HR}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_x & \sin\theta_x \\ 0 & -\sin\theta_x & \cos\theta_x \end{bmatrix} \begin{bmatrix} \cos\theta_y & 0 & -\sin\theta_y \\ 0 & 1 & 0 \\ \sin\theta_y & 0 & \cos\theta_y \end{bmatrix} \begin{bmatrix} \cos\theta_z & \sin\theta_z & 0 \\ -\sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.2}$$

Since θ_x, θ_y and θ_z are assumed to be of order $\epsilon^{3\backslash 2}$, the sine and cosine functions can be approximated as

$$\sin\theta \approx \theta$$

and

$$\cos\theta \approx 1$$

Hence -

$$[T_{HR}] = \begin{bmatrix} 1 & \theta_z & -\theta_y \\ \theta_y\theta_z - \theta_z & 1 & \theta_x \\ \theta_x\theta_z + \theta_y & \theta_y\theta_z - \theta_x & 1 \end{bmatrix} \tag{3.3}$$

3.3 ROTATING HUB FIXED SYSTEM -1

The coordinate system 1 shown in Fig.4, rotates about z_H axis with constant angular speed Ω of the rotor. Its origin is fixed at hub centre O_H. This system

can be obtained by rotating H system by an azimuthal angle ψ_k of the k^{th} blade about z_H axis. The transformation matrix is given as -

$$[T_{1H}] = \begin{bmatrix} \cos\psi_k & \sin\psi_k & 0 \\ -\sin\psi_k & \cos\psi_k & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.4}$$

where, ψ_k is azimuth angle of k^{th} blade

$$\psi_k = \Omega\, t + (k-1)\frac{2\pi}{N_b}$$

3.4 ROTATING SYSTEM -2K

The coordinate system 2k shown in Fig.5 is a blade fixed coordinate system which rotates with k^{th} blade. The origin of the 2K system is at the location of the blade root, which is at a distance $a\hat{e}_{y1} + e_1\hat{e}_{x1}$ from the hub centre. 1-system and 2k-system are parallel.

3.5 PRECONED, ROTATING SYSTEM -3K

The system 3k shown in Fig.6a and Fig.6b also rotates with blade. This system is obtained by rotating 2k -system by an angle $-\beta_p$ (precone angle) about y_{2k} axis. The transformation matrix between 2k and 3k systems is given as -

$$[T_{32}] = \begin{bmatrix} 1 & 0 & \beta_p \\ 0 & 1 & 0 \\ -\beta_p & 0 & 1 \end{bmatrix} \tag{3.5}$$

3.6 PREDROOPED, PRESWEPT, PITCHED, BLADE-FIXED ROTATING SYSTEM -4K

The 4k system shown in Fig.6a and Fig.6b, is blade fixed system with its origin at pitch bearing of the blade .

It may be noted that the pitch axis of the blade is along $\hat{e}_{x3k}$ direction and the blade reference elastic axis is along the $\hat{e}_{x4k}$ direction. During the control pitch input, the elastic axis will move on surface of a cone whose vortex angle is described by the angles β_s and β_d as shown in Fig.6c.

The 4k system is obtained by the following steps -

Step-1 Translating the origin of 3k system by a distance 'e_2' along e_{x3k}.

Step-2 Then rotating the system by an angle $-\beta_s$ (presweep angle) about z_{3k} axis.

Step-3 Then rotating the system by an angle $-\beta_d$ (predroop angle) about y_{3k} axis.

Step-4 Then rotating the system by an angle θ_I (pitch input) about x_{3k} axis.

The transformation matrix is given as -

$$[T_{43}] = \begin{bmatrix} 1 & -(\beta_s \cos\theta_I + \beta_d \sin\theta_I) & (\beta_d \cos\theta_I - \beta_s \sin\theta_I) \\ (\beta_s \cos\theta_I + \beta_d \sin\theta_I) & \cos\theta_I & \sin\theta_I \\ (-\beta_d \cos\theta_I + \beta_s \sin\theta_I) & -\sin\theta_I & \cos\theta_I \end{bmatrix} \tag{3.6}$$

3.7 ROTATING, BLADE-FIXED SYSTEM -5K

The 5k system, shown in Fig.7 is a cross-sectional coordinate system of the k^{tk} blade. In the undeformed state of the blade, both 4k and 5k systems are parallel. But the origin of the 5k system is at a distance x_k from the origin of the 4k system. During elastic deformation of the blade, the 5k system translates and rotates with the cross-section. After the deformation, the origin of the 5k system, from the origin of 4k system, is at the location given by

$$(< n-1 > l_e + x_k + u_k)\hat{e}_{x4k} + v_k\hat{e}_{y4k} + w_k\hat{e}_{x4k}$$

The transformation matrix between 4k and 5k system is obtained following a flap - lag sequence of rotation. The Euler angles are respectively $-\beta_k$ and ξ_k which correspond to the local slope of the deformed blade. The transformation matrix is

given as:

$$[T_{54}] = \begin{bmatrix} \cos\xi_k & \sin\xi_k & 0 \\ -\sin\xi_k & \cos\xi_k & 0 \\ & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\beta_k & 0 & \sin\beta_k \\ 0 & 1 & 0 \\ -\sin\beta_k & 0 & \cos\beta_k \end{bmatrix} \tag{3.7}$$

Since the angle ξ_k and β_k are of the order $O(\epsilon)$, the transformation matrix can be simplified by assuming -

$$\sin\theta \approx \theta$$

and

$$\cos\theta \approx 1$$

Defining the Euler angles in the terms of elastic deformation of the blade

$$\beta_k = w_k' \approx \frac{dw_k}{dx_k}$$

$$\xi_k = v_k' \approx \frac{dv_k}{dx_k}$$

Substituting the above relations in the transformation matrix $[T_{54}]$ yields

$$[T_{54}] = \begin{bmatrix} 1 & v_k' & w_k' \\ -v_k' & 1 & -v_k'w_k' \\ -w_k' & 0 & 1 \end{bmatrix} \tag{3.8}$$

3.8 COORDINATE SYSTEM -6K

The 6k system shown in Fig.8 represents the cross-sectional coordinate system in the deformed configuration of the blade. $\hat{e}_\eta, \hat{e}_\zeta$ represent the directions of the cross-sectional principal axes. 6k-system is obtained by rotating 5k system about $\hat{e}_{x5k}$ through the angle $(\phi_k + \theta_G)$, where θ_G represents the geometric twist angle of the cross-section and ϕ_k represents the elastic twist.

The transformation relation is given as -

$$\begin{Bmatrix} \hat{e}_x \\ \hat{e}_y \\ \hat{e}_z \end{Bmatrix}_{5k} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta_G+\phi_k) & -\sin(\theta_G+\phi_k) \\ 0 & \sin(\theta_G+\phi_k) & \cos(\theta_G+\phi_k) \end{bmatrix} \begin{Bmatrix} \hat{e}_x \\ \hat{e}_\eta \\ \hat{e}_\zeta \end{Bmatrix}_{6k} \tag{3.9}$$

Summary of Coordinate Systems

Symbol	Origin	Coordinate System	Unit vectors
R	Hub centre O_H	Inertial. Fixed at undeformed hub centre 'O_H'	$\hat{e}_{xR}$, $\hat{e}_{yR}$, $\hat{e}_{zR}$
H	Hub centre O_H	Noninertial , body fixed.	$\hat{e}_{xH}$, $\hat{e}_{yH}$, $\hat{e}_{zH}$
1	Hub centre O_H	Rotates with k^{th} blade with angular speed Ω	$\hat{e}_{x1}$, $\hat{e}_{y1}$, $\hat{e}_{z1}$
2k	At point 'A' shown in Fig.6.	Rotates with k^{th} blade. Systems 1 and 2k are parallel.	$\hat{e}_{x2k}$, $\hat{e}_{y2k}$, $\hat{e}_{z2k}$
3k	At point 'A' shown in Fig.6.	Preconed, rotates with k^{th} blade.	$\hat{e}_{x3k}$, $\hat{e}_{y3k}$, $\hat{e}_{z3k}$
4k	out board of pitch bearing	Predrooped, preswept, pitched blade fixed system. Rotates with k^{th} blade.	$\hat{e}_{x4k}$, $\hat{e}_{y4k}$, $\hat{e}_{z4k}$
5k	At elastic axis	Rotates with k^{th} blade. Deformed blade coordinate system.	$\hat{e}_{x5k}$, $\hat{e}_{y5k}$, $\hat{e}_{z5k}$
6k	At elastic axis	Rotates with k^{th} blade. Obtained by rotating 5k system through an angle $(\theta_G + \phi_k)$ about $\hat{e}_{x5k}$	$\hat{e}_{x6k}$, $\hat{e}_{\eta}$, $\hat{e}_{\zeta}$

Chapter 4

KINEMATICS

During operation, the rotor blade undergoes deformation in in-plane bending (lag), out-of-plane bending (flap), torsion and axial modes. In addition, the hub centre has both translational (R_x, R_y, R_z) and rotational ($\theta_x, \theta_y, \theta_z$) perturbational motion. The formulation of inertia operator and aerodynamic operator requires a proper discription of kinematics of the blade motion. In this section, an expression for the absolute velocity vector at any arbitrary point 'p' on the blade is derived.

4.1 POSITION VECTOR OF A POINT

The position vector of any arbitrary point 'p' in the n^{th} finite element of the blade (Fig. 9) with respect to the hub centre O_H, is given by

$$\begin{aligned} \vec{r_p} &= a\hat{e}_{y1} + e_1\hat{e}_{x2k} + e_2\hat{e}_{x3k} + (< n-1 > l_e + x_k + u_k)\hat{e}_{x4k} + v_k\hat{e}_{y4k} \\ &\quad + w_k\hat{e}_{z4k} + \eta\hat{e}_\eta + \zeta\hat{e}_\zeta \end{aligned} \tag{4.1}$$

Transforming all the unit vectors to the 4k - system and neglecting the higher order terms, the position vector can be written as:

$$\begin{aligned}
\vec{r_p} \;=\;& \hat{e_{4x}}l[-a(\beta_s\cos\theta_I-\beta_d\sin\theta_I)+e_1+e_2+(\langle n-1\rangle l_e+x_k+u_k)\\
&+\eta\{-v_k'\cos(\theta_G+\phi_k)-w_k'\sin(\theta_G+\phi_k)\}+\zeta\{v_k'\sin(\theta_G+\phi_k)-w_k'\cos(\theta_G+\phi_k)\}]\\
&+\hat{e_{4y}}l[a\cos\theta_I+e_1(\beta_s\cos\theta_I-\beta_d\sin\theta_I-\beta_p si)+e_2(\beta_s\cos\theta_I-\beta_d\sin\theta_I)+v_k\\
&+\eta\cos(\theta_G+\phi_k)-\zeta\sin(\theta_G+\phi_k)]\\
&+\hat{e_{4z}}l[-a\sin\theta_I-e_1(\beta_d\cos\theta_I+\beta_s\sin\theta_I+\beta_p\cos\theta_I)-e_2(\beta_d\cos\theta_I+\beta_s\sin\theta_I)+w_k\\
&+\eta\{-w_k'v_k'\cos(\theta_G+\phi_k)+\sin(\theta_G+\phi_k)\}+\zeta\{w_k'v_k'+\cos(\theta_G+\phi_k)\}] \qquad (4.2)
\end{aligned}$$

In Equation 4.2 all the length quantities are nondimensionalized with respect to the length of the blade 'l'. For the sake of convenience, the nondimensional length quantities are referred without an overbar $(\bar{\ })$. Equation 4.2 can also be written in symbolic form as

$$\vec{r_p}=l[r_x\hat{e_{x4k}}+r_y\hat{e_{y4k}}+r_z\hat{e_{z4k}}]$$

Differentiating the position vector $\vec{r_p}$ with respect to time

$$\begin{aligned}
\dot{\vec{r_p}} \;=\;& \hat{e}_{4x}l\Omega[a\dot{\theta}_I(\beta_d\cos\theta_I+\beta_s\sin\theta_I)+\dot{u}_k+\eta\{-\dot{v}_k'\cos(\theta_G+\phi_k)\\
&+v_k'\dot{\phi}_k\sin(\theta_G+\phi_k)-\dot{w}_k'\sin(\theta_G+\phi_k)-w_k'\dot{\phi}_k\cos(\theta_G+\phi_k)\}\\
&+\zeta\{\dot{v}_k'\sin(\theta_G+\phi_k)+v_k'\dot{\phi}_k\cos(\theta_G+\phi_k)\\
&-\dot{w}_k'\cos(\theta_G+\phi_k)+w_k'\dot{\phi}_k\sin(\theta_G+\phi_k)\}]\\
&+\hat{e}_{4y}l\Omega[-a\dot{\theta}_I\sin\theta_I-\dot{\theta}_I(e_1+e_2)(\beta_s\sin\theta_I+\beta_d\cos\theta_I)-e_1\beta_p\dot{\theta}_I\cos\theta_I\\
&-\dot{\phi}_k\{\eta\sin(\theta_G+\phi_k)+\zeta\cos(\theta_G+\phi_k)\}+\dot{v}_k]\\
&+\hat{e}_{4z}l\Omega[-a\dot{\theta}_I\cos\theta_I+\dot{\theta}_I(e_1+e_2)(\beta_d\sin\theta_I-\beta_s\cos\theta_I)+e_1\beta_p\dot{\theta}_I\sin\theta_I+\dot{w}_k\\
&+\{\dot{\phi}_k\eta\cos(\theta_G+\phi_k)-\zeta\sin(\theta_G+\phi_k)\}]
\end{aligned} \qquad (4.3)$$

where $(\dot{\ })$ indicates differentiation with respect to the nondimensional time ψ,

$$\psi=\Omega t$$

Note that

$$\frac{d}{dt}(\ ) = \Omega \frac{d}{d\Omega t}(\ ) = \Omega \frac{d}{d\psi}(\ ) = \Omega (\dot{\ })$$

4.2 ANGULAR VELOCITY VECTOR

The angular velocity vector $\vec{\omega}$ of k^{th} blade consists of three components. They are
(i) the constant rotational speed (Ω) of the rotor;
(ii) the rigid body angular velocity ($\vec{\omega}_{rigid}$) of the hub due to perturbational rotation in roll-pitch-yaw (θ_x, θ_y and θ_z)
(iii) the angular velocity contribution due to the rate of change of control pitch input $\Omega\dot{\theta}_I$ to the blade

$$\vec{\omega} = \Omega \hat{e}_{zH} + \vec{\omega}_{rigid} + \Omega \dot{\theta}_I \hat{e}_{x3k} \tag{4.4}$$

where

$$\vec{\omega}_{rigid} = \Omega \dot{\theta}_x \hat{e}_{x1k} + \Omega \dot{\theta}_y \hat{e}_{y1k} + \Omega \dot{\theta}_z \hat{e}_{z1k} \tag{4.5}$$

Transforming all the unit vectors of Equation 4.4 to the 4k-system and neglecting the higher order terms, the angular velocity of k^{th} blade can be written in symbolic form as:

$$\vec{\omega} = \Omega[\omega_x \hat{e}_{x4k} + \omega_y \hat{e}_{y4k} + \omega_z \hat{e}_{z4k}] \tag{4.6}$$

where

$$\begin{aligned}
\omega_x &= [\dot{\theta}_x \cos\psi_k + \dot{\theta}_y \sin\psi_k + \beta_p + \beta_d \cos\theta_I + \beta_s \sin\theta_I + (1 + \beta_d < \sin\theta_I + \cos\theta_I > \\
&\quad + \beta_s < \sin\theta_I - \cos\theta_I >)\dot{\theta}_I] && (4.7) \\
\omega_y &= [(-\dot{\theta}_x \sin\psi_k + \dot{\theta}_y \cos\psi_k)\cos\theta_I + (1 + \dot{\theta}_z)\sin\theta_I \\
&\quad + (1 - \beta_d \sin\theta_I + \beta_s \cos\theta_I)\dot{\theta}_I] && (4.8) \\
\omega_z &= [(1 + \dot{\theta}_z)\cos\theta_I + (1 - \beta_d \cos\theta_I - \beta_s \sin\theta_I)\dot{\theta}_I] && (4.9)
\end{aligned}$$

4.3 VELOCITY AT POINT 'p'

The absolute velocity vector $\vec{V}$, at point 'p' on the deformed beam can be written as:

$$\vec{V} = \vec{V}_H + \dot{\vec{r}}_p + (\vec{\omega} \times \vec{r}) \tag{4.10}$$

where $\vec{V}_H$ is the rigid body perturbational translation of the hub centre O_H which is given as:

$$\vec{V}_H = \Omega\, l[\dot{R}_x \hat{e}_{xR} + \dot{R}_y \hat{e}_{yR} + \dot{R}_z \hat{e}_{zR}] \tag{4.11}$$

Transforming all the unit vectors in 4k -system

$$\vec{V}_H = (V_H)_x \hat{e}_{x4k} + (V_H)_y \hat{e}_{y4k} + (V_H)_z \hat{e}_{z4k} \tag{4.12}$$

where

$$\begin{aligned}
(V_H)_x &= \Omega\, l\dot{R}_x\{\cos\psi_k - \sin\psi_k(-\beta_s\cos\theta_I + \beta_d\sin\,\theta_I)\} \\
&\quad +\Omega\, l\dot{R}_y\{\sin\psi_k + \cos\,\psi_k(-\beta_s\,\cos\theta_I + \beta_d\sin\theta_I)\} \\
&\quad +\Omega\, l\dot{R}_z\{\beta_p + \beta_d\cos\theta_I + \beta_s\sin\,\theta_I\} && (4.13) \\
(V_H)_y &= \Omega\, l\dot{R}_x\{\cos\psi_k(\beta_s\cos\theta_I - \beta_d\sin\,\theta_I) - \sin\psi_k\cos\theta_I\} \\
&\quad +\Omega\, l\dot{R}_y\{\sin\psi_k(\beta_s\cos\theta_I - \beta_d\sin\theta_I) + \cos\psi_k\cos\theta_I\} \\
&\quad +\Omega\, l\dot{R}_z\sin\theta_I && (4.14) \\
(V_H)_z &= -\Omega\, l\dot{R}_x\{\cos\psi_k(\beta_p + \beta_d\cos\theta_I + \beta_s\sin\theta_I) + \sin\,\psi_k si\} \\
&\quad -\Omega\, l\dot{R}_y\{\sin\psi_k(\beta_p + \beta_d\cos\theta_I + \beta_s\sin\theta_I) - \cos\,\psi_k si\} \\
&\quad +\Omega\, l\dot{R}_z\cos\theta_I && (4.15)
\end{aligned}$$

and

$$(\vec{\omega} \times \vec{r}) = \begin{vmatrix} \hat{e}_{x4k} & \hat{e}_{y4k} & \hat{e}_{z4k} \\ \omega_x & \omega_y & \omega_z \\ r_x & r_y & r_z \end{vmatrix} \tag{4.16}$$

Where r_x, r_y, r_z and ω_x, ω_y, ω_z are the x, y and z components of $\vec{r}_p$ and $\vec{\omega}$ respectively. Substituting various quantities in Equation 4.10 from Equations 4.2, 4.3, 4.7-4.9 and

4.12-4.16, the velocity at point p is obtained. This can be written in symbolic form as:

$$\vec{V} = \Omega\, l[V_x \hat{e}_{x4k} + V_y \hat{e}_{y4k} + V_z \hat{e}_{z4k}] \tag{4.17}$$

The components of the velocity vector are given below.

$$
\begin{aligned}
V_x &= (V_H)_x + (\dot{r}_p)_x + (\omega_y r_z - \omega_z r_y) \\
&= \dot{R}_x\{\cos\psi_k - \sin\psi_k(-\beta_s\cos\theta_I + \beta_d\sin\theta_I)\} \\
&\quad + \dot{R}_y\{\sin\psi_k + \cos\psi_k(-\beta_s\cos\theta_I + \beta_d\sin\theta_I)\} \\
&\quad + \dot{R}_z\{\beta_p + \beta_d\cos\theta_I + \beta_s\sin\theta_I\} \\
&\quad + a\dot{\theta}_I(\beta_d\cos\theta_I + \beta_s\sin\theta_I) + \dot{u}_k + \eta\{-\dot{v}_k'\cos(\theta_G+\phi_k) \\
&\quad + v_k'\dot{\phi}_k\sin(\theta_G+\phi_k) - \dot{w}_k'\sin(\theta_G+\phi_k) - w_k'\dot{\phi}_k\cos(\theta_G+\phi_k)\} \\
&\quad + \zeta\{\dot{v}_k'\sin(\theta_G+\phi_k) + v_k'\dot{\phi}_k\cos(\theta_G+\phi_k) \\
&\quad - \dot{w}_k'\cos(\theta_G+\phi_k) + w_k'\dot{\phi}_k\sin(\theta_G+\phi_k)\} \\
&\quad + \dot{\theta}_x\cos\theta_I[\sin\psi_k\{-w_k + \eta\sin(\theta_G+\phi_k) - \zeta\cos(\theta_G+\phi_k)\}] \\
&\quad - \dot{\theta}_y\cos\theta_I[\cos\psi_k\{-w_k + \eta\sin(\theta_G+\phi_k) - \zeta\cos(\theta_G+\phi_k)\} \\
&\quad - \dot{\theta}_z\sin\theta_I\{-w_k + \eta\sin(\theta_G+\phi_k) - \zeta\cos(\theta_G+\phi_k)\}] \\
&\quad - \dot{\theta}_z\cos\theta_I[a\cos\theta_I + v_k + \eta\cos(\theta_G+\phi_k) + \zeta\sin(\theta_G+\phi_k)] \\
&\quad - \dot{\theta}_I[\{-w_k + \eta\sin(\theta_G+\phi_k) - \zeta\cos(\theta_G+\phi_k) + e_1 + e_2\}\{\beta_s\cos\theta_I - \beta_d\sin\theta_I\} \\
&\quad + e_1\beta_p\sin\theta_I + \{e_1 + e_2 - a\cos\theta_I - v_k - \eta\cos(\theta_G+\phi_k) - \zeta\sin(\theta_G+\phi_k)\} \\
&\quad \{\beta_d\cos\theta_I + \beta_s\sin\theta_I\} + \eta\{\cos(\theta_G+\phi_k) - \sin(\theta_G+\phi_k)\} \\
&\quad + \zeta\{\sin(\theta_G+\phi_k) + \cos(\theta_G+\phi_k)\} + v_k + w_k + a(\cos\theta_I - \sin\theta_I)] \\
&\quad - \cos\theta_I[a\cos\theta_I + (e_1+e_2)(\beta_s\cos\theta_I - \beta_d\sin\theta_I) + v_k \\
&\quad + \eta\cos(\theta_G+\phi_k) + \zeta\sin(\theta_G+\phi_k)] \\
&\quad - \sin\theta_I[(e_1+e_2)(\beta_d\cos\theta_I - \beta_s\sin\theta_I) + w_k \\
&\quad - \eta\sin(\theta_G+\phi_k) + \zeta\cos(\theta_G+\phi_k)]
\end{aligned}
\tag{4.18}
$$

$$
\begin{aligned}
V_y &= (V_H)_y + (\dot{r}_p)_y - (\omega_x r_z - \omega_z r_x) \\
&= \dot{R}_x[\cos\psi_k(\beta_s\cos\theta_I - \beta_d\sin\theta_I) - \sin\psi_k\cos\theta_I] \\
&\quad +\dot{R}_y[\sin\psi_k(\beta_s\cos\theta_I - \beta_d\sin\theta_I) + \cos\psi_k\cos\theta_I] \\
&\quad -\dot{R}_z\sin\theta_I - a\dot{\theta}_I\sin\theta_I - e_1\dot{\theta}_I\beta_p\cos\theta_I \\
&\quad -\{\dot{\theta}_I(e_1+e_2)(\beta_s\sin\theta_I + \beta_d\cos\theta_I)\} \\
&\quad -\dot{\phi}_k\{\eta\sin(\theta_G+\phi_k) + \zeta\cos(\theta_G+\phi_k)\} + \dot{v}_k \\
&\quad +\dot{\theta}_x[\cos\psi_k\{-w_k + \eta\sin(\theta_G+\phi_k) - \zeta\cos(\theta_G+\phi_k)\}] \\
&\quad -\dot{\theta}_y[\sin\psi_k\{-w_k + \eta\sin(\theta_G+\phi_k) - \zeta\cos(\theta_G+\phi_k)\}] \\
&\quad -\dot{\theta}_z\cos\theta_I[e_1 + e_2 + <n-1> l_e + x_k + u_k - v'_k\{\eta\cos(\theta_G+\phi_k) - \zeta\sin(\theta_G+\phi_k)\} \\
&\quad -w'_k\{\eta\sin(\theta_G+\phi_k) - \zeta\cos(\theta_G+\phi_k)\}] \\
&\quad -\dot{\theta}_I[\{bd(\sin\theta_I + \cos\theta_I) + \beta_s(\sin\theta_I - \cos\theta_I) + 1\}\{w_k - \eta\sin(\theta_G+\phi_k) + \zeta\cos(\theta_G+\phi_k)\} \\
&\quad +(\beta_d\cos\theta_I + \beta_s\sin\theta_I)\{2(e_1+e_2) + <n-1> l_e + x_k + u_k - v'_k(\eta\cos<\theta_G+\phi_k> \\
&\quad -\zeta\sin<\theta_G+\phi_k>)\} - w'_k(\eta\sin<\theta_G+\phi_k> + \zeta\cos<\theta_G+\phi_k>)\} \\
&\quad -v'_k\{\eta\cos(\theta_G+\phi_k) - \zeta\sin(\theta_G+\phi_k)\} - w'_k\{\eta\sin(\theta_G+\phi_k) + \zeta\cos(\theta_G+\phi_k)\} \\
&\quad -v'_k w'_k\{\eta\cos(\theta_G+\phi_k) - \zeta\sin(\theta_G+\phi_k)\} + a(\beta_s\cos\theta_I - \beta_d\sin\theta_I) \\
&\quad -(e_1+e_2) - <n-1> l_e - x_k - u_k] \\
&\quad -[(\beta_p + \beta_d\cos\theta_I + \beta_s\sin\theta_I)\{w_k - \eta\sin(\theta_G+\phi_k) + \zeta\cos(\theta_G+\phi_k)\} \\
&\quad +\cos\theta_I\{-v'_k\{\eta\cos(\theta_G+\phi_k) - \zeta\sin(\theta_G+\phi_k)\} \\
&\quad -w'_k\{\eta\sin(\theta_G+\phi_k) + \zeta\cos(\theta_G+\phi_k)\} \\
&\quad -a(\beta_s\cos\theta_I - \beta_d\sin\theta_I) + e_1 + e_2 + <n-1> l_e + x_k + u_k\}] \qquad (4.19)
\end{aligned}
$$

$$
\begin{aligned}
V_z &= (V_H)_z + (\dot{r}_p)_z + (\omega_x r_y - \omega_y r_x) \\
&= -\dot{R}_x[\cos\psi_k(\beta_p + \beta_d \cos\theta_I + \beta_s \sin\theta_I) + \sin\psi_k \sin\theta_I] \\
&\quad -\dot{R}_y[\sin\psi_k(\beta_p + \beta_d \cos\theta_I + \beta_s \sin\theta_I) - \cos\psi_k \sin\theta_I] \\
&\quad +\dot{R}_z \cos\theta_I - \dot{\theta}_I(a\cos\theta_I - e_1\beta_p \sin\theta_I) \\
&\quad +\dot{\theta}_I(e_1 + e_2)(\beta_d \sin\theta_I - \beta_s \cos\theta_I) + \{\dot{\phi}_k \eta \cos(\theta_G + \phi_k) - \zeta \sin(\theta_G + \phi_k)\} + \dot{w}_k \\
&\quad +\dot{\theta}_x[\cos\psi_k\{a\cos\theta_I + v_k + \eta\cos(\theta_G + \phi_k) + \zeta\sin(\theta_G + \phi_k)\} + \sin\psi_k \cos\theta_I\{e_1 + e_2 \\
&\quad + < n-1 > l_e + x_k + u_k - v_k'(\eta\cos < \theta_G + \phi_k > -\zeta \sin < \theta_G + \phi_k >) \\
&\quad -w_k'(\eta \sin < \theta_G + \phi_k > +\zeta\cos < \theta_G + \phi_k >)\} \\
&\quad +\dot{\theta}_y[\sin\psi_k\{a\cos\theta_I + v_k + \eta\cos(\theta_G + \phi_k) + \zeta\sin(\theta_G + \phi_k)\} - \cos\psi_k \cos\theta_I\{e_1 + e_2 \\
&\quad + < n-1 > l_e + x_k + u_k - v_k'(\eta\cos < \theta_G + \phi_k > -\zeta \sin < \theta_G + \phi_k >) \\
&\quad -w_k'(\eta \sin < \theta_G + \phi_k > +\zeta\cos < \theta_G + \phi_k >)\} \\
&\quad +\dot{\theta}_I[\{bd(\sin\theta_I + \cos\theta_I) + \beta_s(\sin\theta_I - \cos\theta_I) + 1\}\{a\cos\theta_I + v_k + \eta\cos(\theta_G + \phi_k) \\
&\quad +\zeta\sin(\theta_G + \phi_k)\} - e_1\beta_p \sin\theta_I \\
&\quad -(\beta_s\cos\theta_I - \beta_d \sin\theta_I + 1)\{< n-1 > l_e + x_k + u_k - v_k'(\eta\cos < \theta_G + \phi_k > \\
&\quad -\zeta\sin < \theta_G + \phi_k >) - w_k'(\eta\sin < \theta_G + \phi_k > +\zeta\cos < \theta_G + \phi_k >)\} \\
&\quad +a(\beta_s\cos\theta_I - \beta_d\sin\theta_I) + (e_1 + e_2)] \\
&\quad +[(\beta_p + \beta_d\cos\theta_I + \beta_s\sin\theta_I)\{a + \eta\cos(\theta_G + \phi_k) + \zeta\sin(\theta_G + \phi_k)\}] \\
&\quad +(1 + \dot{\theta}_z)\sin\theta_I[e_1 + e_2 + < n-1 > l_e + x_k + u_k \\
&\quad -v_k'(\eta\cos < \theta_G + \phi_k > -\zeta\sin < \theta_G + \phi_k >) \\
&\quad -w_k'(\eta\sin < \theta_G + \phi_k > +\zeta\cos < \theta_G + \phi_k >) - a\dot{\theta}_I(\beta_s\cos\theta_I - \beta_d\sin\theta_I)] \qquad (4.20)
\end{aligned}
$$

Chapter 5

EQUATIONS OF MOTION FOR ROTOR BLADE

The equations of motion for the rotor blade can be derived using Hamilton's principle. The mathematical form of Hamilton's principle is stated as follows:

$$\int_{t_1}^{t_2} (\delta U - \delta T - \delta W_e) dt = 0 \tag{5.1}$$

where U is the strain energy; T is kinetic energy; W_e is the work done by the non-conservative external loads. Equation 4.1 is an integral equation which states that the total dynamic potential, $(U - T - W_e)$, is an extremum over the time interval ; $t_1 \leq t \leq t_2$.

In this chapter, the expressions for the variation of kinetic energy and strain energy of the rotor blade are derived.

5.1 KINETIC ENERGY OF BLADE

The total kinetic energy of the beam, T, is defined as :

$$\begin{aligned} T &= \frac{1}{2} \int\int\int_V \rho \vec{V}.\vec{V}\, d\,Vol \\ &= \frac{1}{2} \int_0^l \int\int_A \rho \vec{V}.\vec{V}\, d\eta\, d\zeta\, dx \end{aligned} \tag{5.2}$$

where $\vec{V}$ is the velocity of an arbitrary point 'p' on the blade cross section with respect to the inertial reference frame. The variation of kinetic energy, δT can be written as:

$$\delta T = \int_0^l \int\int_A \rho \vec{V}.\delta\vec{V}\, d\eta\, d\zeta\, dx \tag{5.3}$$

Substituting for the velocity $\vec{V}$ from Equation 4.17 and integrating δ T by parts with respect to time, yields

$$\delta T = \int_0^l \int\int_A \rho[Z_u\,\delta u + Z_v\,\delta v + Z_w\,\delta w + Z_v'\,\delta v' + Z_w'\,\delta w' + Z_\phi\,\delta\phi] d\eta\, d\zeta\, dx \tag{5.4}$$

where the terms Z_u, Z_v, Z_w, Z_v', Z_w', Z_ϕ are the coefficients of δu, δv, δw, $\delta v'$, $\delta w'$, $\delta\phi$ in the variation of kinetic energy expression.

Integration of the expression over the cross-section yields:

$$\delta T = m\Omega^2 l^3 \int_0^1 [\bar{Z}_u\,\delta u + \bar{Z}_v\,\delta v + \bar{Z}_w\,\delta w + \bar{Z}_v'\,\delta v' + \bar{Z}_w'\,\delta w' + \bar{Z}_\phi\,\delta\phi] dx \tag{5.5}$$

After eliminating the higher order terms, the expressions for $\bar{Z}_u$, $\bar{Z}_v$, $\bar{Z}_w$, $\bar{Z}_v'$, $\bar{Z}_w'$, $\bar{Z}_\phi$ are obtained. The detailed expressions of these coefficients are given in Appendix- A.

5.2 STRAIN ENERGY OF BLADE

The formulation of strain energy outlined in this section essentially follows the procedure given in Refs. 23 and 28.

5.2.1 Strain Energy

The strain energy of a beam element is given by

$$U = \frac{1}{2}E_o l^3 \int_0^{l_e} \int\int \begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{x\eta} \\ \epsilon_{x\zeta} \end{Bmatrix}^T \begin{Bmatrix} \sigma_{xx} \\ \sigma_{x\eta} \\ \sigma_{x\zeta} \end{Bmatrix} d\eta d\zeta dx \tag{5.6}$$

5.2.2 Explicit Strain-Displacement Relations

The expressions for non zero strain components written in terms of u,v,w and ϕ are given as [23]

$$\begin{aligned}
\epsilon_{xx} &= \underline{u_x + \frac{1}{2}v_x^2 + \frac{1}{2}w_x^2} + \frac{1}{2}(\eta^2+\zeta^2)\phi_x^2 + \alpha_x\Psi + \alpha\tau_o(\zeta\Psi_\eta - \eta\Psi_\zeta) \\
&\quad -[\eta\cos(\theta_g+\phi) - \zeta\sin(\theta_g+\phi)]v_{xx} - [\eta\sin(\theta_g+\phi) + \zeta\cos(\theta_g+\phi)]w_{xx} \\
&\quad +\eta(\bar{\gamma}_{x\eta,x} - \tau_o\bar{\gamma}_{x\zeta}) + \zeta(\bar{\gamma}_{x\zeta,x} + \tau_o\bar{\gamma}_{x\eta}) \\
\epsilon_{x\eta} &= \bar{\gamma}_{x\eta} + \alpha\Psi_\eta - \zeta(\phi_x + \phi_o) \\
\epsilon_{x\zeta} &= \bar{\gamma}_{x\zeta} + \alpha\Psi_\zeta + \eta(\phi_x + \phi_o)
\end{aligned}$$

where

$$\phi_o = (v_{xx}\cos\theta_g + w_{xx}\sin\theta_g)(-v_x\sin\theta_g + w_x\cos\theta_g)$$

The underlined term in ϵ_{xx} represents the axial strain at the elastic axis. These strain expressions can be simplified using the following assumptions:

- The transverse shear at the elastic axis is assumed to be zero.

- The warping amplitude α is assumed to be equal to $-\phi_x$.

The simplified strain components can be written as

$$\begin{aligned}
\epsilon_{xx} &= u_x + \frac{1}{2}v_x^2 + \frac{1}{2}w_x^2 + \frac{1}{2}(\eta^2+\zeta^2)\phi_x^2 - \Psi\phi_{xx} - [\tau_o(\zeta\Psi_\eta - \eta\Psi_\zeta)]\phi_x \\
&\quad -[\eta\cos(\theta_g+\phi) - \zeta\sin(\theta_g+\phi)]v_{xx} - [\eta\sin(\theta_g+\phi) + \zeta\cos(\theta_g+\phi)]w_{xx}
\end{aligned}$$

$$\epsilon_{x\eta} = -(\Psi_\eta + \zeta)\phi_x - \zeta\phi_o$$

$$\epsilon_{x\zeta} = -(\Psi_\zeta - \eta)\phi_x + \eta\phi_o$$

$$\phi_o = (v_{xx}\cos\theta_g + w_{xx}\sin\theta_g)(-v_x\sin\theta_g + w_x\cos\theta_g)$$

5.2.3 Stress-Strain Relations

The stress-strain relationship is given by the following equation:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{x\eta} \\ \sigma_{x\zeta} \end{Bmatrix} = \begin{bmatrix} E & 0 & 0 \\ 0 & G & 0 \\ 0 & 0 & G \end{bmatrix} \begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{x\eta} \\ \epsilon_{x\zeta} \end{Bmatrix} \tag{5.7}$$

5.2.4 Strain Energy Variation

The variation of strain energy is given by

$$\delta U = E_o l^3 \int_0^{l_e} \int\int \begin{Bmatrix} \delta\epsilon_{xx} \\ \delta\epsilon_{x\eta} \\ \delta\epsilon_{x\zeta} \end{Bmatrix}^T \begin{Bmatrix} \sigma_{xx} \\ \sigma_{x\eta} \\ \sigma_{x\zeta} \end{Bmatrix} d\eta d\zeta dx \tag{5.8}$$

The variation of the strain components is given as:

$$\begin{aligned}
\delta\epsilon_{xx} &= \delta u_x + v_x\delta v_x + w_x\delta w_x + (\eta^2+\zeta^2)\phi_x\delta\phi_x - \Psi\delta\phi_{xx} - [\tau_o(\zeta\Psi_\eta - \eta\Psi_\zeta)]\delta\phi_x \\
&\quad -(\eta\cos\theta_g - \zeta\sin\theta_g)(\delta v_{xx} + \phi\delta w_{xx} + w_{xx}\delta\phi)
\end{aligned}$$

$$-(\eta \sin\theta_g + \zeta \cos\theta_g)(\delta w_{xx} - \phi \delta v_{xx} - v_{xx}\delta\phi)$$

$$\delta\epsilon_{x\eta} = -(\Psi_\eta + \zeta)\delta\phi_x - \zeta\delta\phi_o$$

$$\delta\epsilon_{x\zeta} = -(\Psi_\zeta - \eta)\delta\phi_x + \eta\delta\phi_o$$

where

$$\begin{aligned}\delta\phi_o &= (-\delta v_x \sin\theta_g + \delta w_x \cos\theta_g)(v_{xx}\cos\theta_g + w_{xx}\sin\theta_g) \\ &\quad +(-v_x \sin\theta_g + w_x\cos\theta_g)(\delta v_{xx}\cos\theta_g + \delta w_{xx}\sin\theta_g)\end{aligned}$$

It is assumed that the variations of strain components are of the same order as the corresponding strain components. Substituting the above expressions in the strain energy variation and integrating over the cross-section, the variation in strain energy for a beam element is obtained. The expression for δU, in terms of stress and moment resultants, is given as:

$$\begin{aligned}\delta U &= E_o l^3 \int_0^{l_i} \{\bar{V}_x(\delta u_x + v_x\delta v_x + w_x\delta w_x) \\ &\quad +(-\tau_o \bar{P}_x{}' - \bar{R}_x + \bar{S}_x + \bar{T}_x\phi_x)\delta\phi_x + \bar{S}_x{}'\delta\phi_o - \bar{P}_x\delta\phi_{xx} \\ &\quad +[v_{xx}(\bar{M}_\eta{}'\cos\theta_g - \bar{M}_\zeta{}'\sin\theta_g) + w_{xx}(\bar{M}_\eta{}'\sin\theta_g + \bar{M}_\zeta{}'\cos\theta_g)]\delta\phi \\ &\quad +[(\bar{M}_\eta\sin\theta_g + \bar{M}_\zeta\cos\theta_g) + \phi(\bar{M}_\eta{}'\cos\theta_g - \bar{M}_\zeta{}'\sin\theta_g)]\delta v_{xx} \\ &\quad +[(-\bar{M}_\eta\cos\theta_g + \bar{M}_\zeta\sin\theta_g) + \phi(\bar{M}_\eta{}'\sin\theta_g + \bar{M}_\zeta{}'\cos\theta_g)]\delta w_{xx}\}dx\end{aligned} \tag{5.9}$$

Stress and Moment Resultants are defined as:

$$\begin{aligned}\bar{V}_x &= \int\int (E\epsilon_{xx})d\eta d\zeta \\ &= EA[u_x + \frac{1}{2}v_x^2 + \frac{1}{2}w_x^2] \\ &\quad +\frac{1}{2}EAC_o[\phi_x^2] - EAD_o[\phi_{xx}] - \tau_o EAD_o'[\phi_x] \\ &\quad -(\overline{EA\eta_a} - \phi\overline{EA\zeta_a})[v_{xx}] - (\overline{EA\zeta_a} + \phi\overline{EA\eta_a})[w_{xx}]\end{aligned} \tag{5.10}$$

$$
\begin{aligned}
\bar{M}_\eta &= \int\int(\zeta E\epsilon_{xx})d\eta d\zeta \\
&= EA\zeta_a[u_x + \frac{1}{2}v_x^2 + \frac{1}{2}w_x^2] \\
&\quad +\frac{1}{2}EAC_2[\phi_x^2] - EAD_2[\phi_{xx}] - \tau_o EAD_2'[\phi_x] \\
&\quad -(\overline{EI_{\eta\zeta}} - \phi\overline{EI_{\eta\eta}})[v_{xx}] - (\overline{EI_{\eta\eta}} + \phi\overline{EI_{\eta\zeta}})[w_{xx}] \qquad (5.11)
\end{aligned}
$$

$$
\begin{aligned}
\bar{M}_\eta' &= \int\int(\zeta E\epsilon_{xx})d\eta d\zeta \\
&= EA\zeta_a[u_x + \frac{1}{2}v_x^2 + \frac{1}{2}w_x^2] \\
&\quad -EAD_2[\phi_{xx}] - \tau_o EAD_2'[\phi_x] \\
&\quad -(\overline{EI_{\eta\zeta}})[v_{xx}] - (\overline{EI_{\eta\eta}})[w_{xx}] \qquad (5.12)
\end{aligned}
$$

$$
\begin{aligned}
\bar{M}_\zeta &= \int\int(-\eta E\epsilon_{xx})d\eta d\zeta \\
&= -EA\eta_a[u_x + \frac{1}{2}v_x^2 + \frac{1}{2}w_x^2] \\
&\quad -\frac{1}{2}EAC_1[\phi_x^2] + EAD_1[\phi_{xx}] + \tau_o EAD_1'[\phi_x] \\
&\quad +(\overline{EI_{\zeta\zeta}} - \phi\overline{EI_{\zeta\eta}})[v_{xx}] + (\overline{EI_{\zeta\eta}} + \phi\overline{EI_{\zeta\zeta}})[w_{xx}] \qquad (5.13)
\end{aligned}
$$

$$
\begin{aligned}
\bar{M}_\zeta' &= \int\int(-\eta E\epsilon_{xx})d\eta d\zeta \\
&= -EA\eta_a[u_x + \frac{1}{2}v_x^2 + \frac{1}{2}w_x^2] \\
&\quad +EAD_1[\phi_{xx}] + \tau_o EAD_1'[\phi_x] \\
&\quad +(\overline{EI_{\zeta\zeta}})[v_{xx}] + (\overline{EI_{\zeta\eta}})[w_{xx}] \qquad (5.14)
\end{aligned}
$$

$$
\begin{aligned}
\bar{P}_x &= \int\int(\Psi E\epsilon_{xx})d\eta d\zeta \\
&= EAD_o[u_x + \frac{1}{2}v_x^2 + \frac{1}{2}w_x^2] \\
&\quad +\frac{1}{2}EAD_4[\phi_x^2] - EAD_3[\phi_{xx}] - \tau_o EAD_5[\phi_x]
\end{aligned}
$$

$$-(\overline{EAD_1} - \phi\overline{EAD_2})[v_{xx}] - (\overline{EAD_2} + \phi\overline{EAD_1})[w_{xx}] \tag{5.15}$$

$$\begin{aligned}\bar{P}_x' &= \int\int[(\zeta\Psi_\eta - \eta\Psi_\zeta)E\epsilon_{xx}]d\eta d\zeta \\ &= EAD_o'[u_x + \frac{1}{2}v_x^2 + \frac{1}{2}w_x^2] \\ &\quad + \frac{1}{2}EAD_4'[\phi_x^2] - EAD_5[\phi_{xx}] - \tau_o EAD_3'[\phi_x] \\ &\quad -(\overline{EAD_1}' - \phi\overline{EAD_2}')[v_{xx}] - (\overline{EAD_2}' + \phi\overline{EAD_1}')[w_{xx}]\end{aligned} \tag{5.16}$$

$$\begin{aligned}\bar{T}_x &= \int\int[(\eta^2 + \zeta^2)E\epsilon_{xx}]d\eta d\zeta \\ &= EAC_o[u_x + \frac{1}{2}v_x^2 + \frac{1}{2}w_x^2] \\ &\quad + \frac{1}{2}EAC_3[\phi_x^2] - EAD_4[\phi_{xx}] - \tau_o EAD_4'[\phi_x] \\ &\quad -(\overline{EAC_1}' - \phi\overline{EAC_2}')[v_{xx}] - (\overline{EAC_2}' + \phi\overline{EAC_1}')[w_{xx}]\end{aligned} \tag{5.17}$$

$$\begin{aligned}\bar{R}_x &= \int\int[(\Psi_\eta\epsilon_{x\eta} - \Psi_\zeta\epsilon_{x\zeta})G]d\eta d\zeta \\ &= (GJ_1 - GJ_2)[\phi_x] + GJ_1[\phi_o]\end{aligned} \tag{5.18}$$

$$\begin{aligned}\bar{S}_x &= \int\int[(\eta\epsilon_{x\zeta} - \zeta\epsilon_{x\eta})G]d\eta d\zeta \\ &= (GJ_o - GJ_1)[\phi_x] + GJ_o[\phi_o]\end{aligned} \tag{5.19}$$

$$\begin{aligned}\bar{S}_x{}' &= \int\int[(\eta\epsilon_{x\zeta} - \zeta\epsilon_{x\eta})G]d\eta d\zeta \\ &= (GJ_o - GJ_1)[\phi_x]\end{aligned} \tag{5.20}$$

In the above expressions, the moment resultants $\bar{M_\eta}', \bar{M_\zeta}'$ and $\bar{S_x}'$ have the same definitions as $\bar{M_\eta}, \bar{M_\zeta}$ and $\bar{S_x}$ respectively. But higher order terms are neglected in defining $\bar{M_\eta}'$, $\bar{M_\zeta}'$ and $\bar{S_x}'$, since these expressions are coupled with terms of order ϵ^2. Whereas $\bar{M_\eta}, \bar{M_\zeta}$ and $\bar{S_x}$ are coupled with terms of order ϵ. The cross-sectional

integrals associated with the strain energy variation are defined as:

$$
\begin{aligned}
EA &= \int\int [E]d\eta d\zeta \\
EA\eta_a &= \int\int [E\eta]d\eta d\zeta \\
EA\zeta_a &= \int\int [E\zeta]d\eta d\zeta \\
EI_{\eta\eta} &= \int\int [E\zeta^2]d\eta d\zeta \\
EI_{\zeta\zeta} &= \int\int [E\eta^2]d\eta d\zeta \\
EI_{\eta\zeta} &= \int\int [E\eta\zeta]d\eta d\zeta \\
EAC_o &= \int\int [E(\eta^2+\zeta^2)]d\eta d\zeta \\
EAC_1 &= \int\int [E\eta(\eta^2+\zeta^2)]d\eta d\zeta \\
EAC_2 &= \int\int [E\zeta(\eta^2+\zeta^2)]d\eta d\zeta \\
EAC_3 &= \int\int [E(\eta^2+\zeta^2)^2]d\eta d\zeta \\
EAD_o &= \int\int [E\Psi]d\eta d\zeta \\
EAD_1 &= \int\int [E\eta\Psi]d\eta d\zeta \\
EAD_2 &= \int\int [E\zeta\Psi]d\eta d\zeta \\
EAD_3 &= \int\int [E\Psi^2]d\eta d\zeta \\
EAD_4 &= \int\int [E(\eta^2+\zeta^2)\Psi]d\eta d\zeta \\
EAD_5 &= \int\int [E\Psi(\zeta\Psi_\eta-\eta\Psi_\zeta)]d\eta d\zeta \\
EAD_o' &= \int\int [E(\zeta\Psi_\eta-\eta\Psi_\zeta)]d\eta d\zeta \\
EAD_1' &= \int\int [E\eta(\zeta\Psi_\eta-\eta\Psi_\zeta)]d\eta d\zeta \\
EAD_2' &= \int\int [E\zeta(\zeta\Psi_\eta-\eta\Psi_\zeta)]d\eta d\zeta \\
EAD_3' &= \int\int [E(\zeta\Psi_\eta-\eta\Psi_\zeta)^2]d\eta d\zeta \\
EAD_4' &= \int\int [E(\eta^2+\zeta^2)(\zeta\Psi_\eta-\eta\Psi_\zeta)]d\eta d\zeta \\
GJ_o &= \int\int [G(\eta^2+\zeta^2)]d\eta d\zeta
\end{aligned}
$$

$$\begin{aligned}
GJ_1 &= \int\int [G(\zeta\Psi_\eta - \eta\Psi_\zeta)] d\eta d\zeta \\
GJ_2 &= \int\int [G(\Psi_\eta^2 + \Psi_\zeta^2)] d\eta d\zeta
\end{aligned}$$

Certain additional terms are defined as:

$$\begin{aligned}
\overline{EA\eta_a} &= EA\eta_a \cos\theta_g - EA\zeta_a \sin\theta_g \\
\overline{EA\zeta_a} &= EA\eta_a \sin\theta_g + EA\zeta_a \cos\theta_g \\
\overline{EAC_1} &= EAC_1 \cos\theta_g - EAC_2 \sin\theta_g \\
\overline{EAC_2} &= EAC_1 \sin\theta_g + EAC_2 \cos\theta_g \\
\overline{EAD_1} &= EAD_1 \cos\theta_g - EAD_2 \sin\theta_g \\
\overline{EAD_2} &= EAD_1 \sin\theta_g + EAD_2 \cos\theta_g \\
\overline{EAD_1}' &= EAD_1' \cos\theta_g - EAD_2' \sin\theta_g \\
\overline{EAD_2}' &= EAD_1' \sin\theta_g + EAD_2' \cos\theta_g \\
\overline{EI_{\eta\zeta}} &= EI_{\eta\zeta} \cos\theta_g - EI_{\eta\eta} \sin\theta_g \\
\overline{EI_{\eta\eta}} &= EI_{\eta\zeta} \sin\theta_g + EI_{\eta\eta} \cos\theta_g \\
\overline{EI_{\zeta\zeta}} &= EI_{\zeta\zeta} \cos\theta_g - EI_{\zeta\eta} \sin\theta_g \\
\overline{EI_{\zeta\eta}} &= EI_{\zeta\zeta} \sin\theta_g + EI_{\zeta\eta} \cos\theta_g \\
GJ &= GJ_o - 2GJ_1 + GJ_2
\end{aligned}$$

Chapter 6

FORMULATION OF ELEMENT MATRICES ASSOCIATED WITH KINETIC AND STRAIN ENERGY VARIATION

6.1 FINITE ELEMENT DISCRETIZATION

The variational expressions associated with the kinetic and potential energy of the rotor blades are nonlinear. The unknowns are the deformation functions u, v, w, and ϕ. These are dependent on both space and time. The spatial dependence is eliminated using a Rayleigh Ritz finite element formulation. The blade is divided into subregions(finite elements) as shown in Fig. 9, and total dynamic potential is calculated for each subregion. By applying Hamilton's principle to each subregion, a discretized form of the equations of motion can be obtained. In this developement, each subregion is modelled by a straight beam type finite element. These beam elements are located along the elastic axis (line of shear centers) of the blade.

The discretized form of Hamilton's principle is written as:

$$\int_{t_1}^{t_2} \sum_{i=1}^{N} (\delta U_i - \delta T_i - \delta W_{ei}) dt = 0 \tag{6.1}$$

where:

N = Total number of finite elements in the model.

δU_i = Variation of the strain energy in the i^{th} element.

δT_i = Variation of the kinetic energy in the i^{th} element.

δW_{ei} = Virtual work of the external loads in i^{th} element.

Assume that the four unknown displacement functions of the beam finite element are expressed in the following form.

$$\begin{bmatrix} v \\ w \\ \phi \\ u \end{bmatrix} = \begin{bmatrix} \{\Phi_v\}^T & 0 & 0 & 0 \\ 0 & \{\Phi_w\}^T & 0 & 0 \\ 0 & 0 & \{\Phi_\phi\}^T & 0 \\ 0 & 0 & 0 & \{\Phi_u\}^T \end{bmatrix} \begin{bmatrix} \{V\} \\ \{W\} \\ \{\Phi\} \\ \{U\} \end{bmatrix} \tag{6.2}$$

where $\{\Phi_v\}, \{\Phi_w\}, \{\Phi_\phi\}, \{\Phi_u\}$ are space dependent interpolation functions; and $\{V\}, \{W\}, \{\Phi\}, \{U\}$ are time dependent nodal values of v, w, ϕ, u, respectively.

$$\{V\} = \begin{Bmatrix} v_1 \\ v_1' \\ v_2 \\ v_2' \end{Bmatrix}; \{W\} = \begin{Bmatrix} w_1 \\ w_1' \\ w_2 \\ w_2' \end{Bmatrix}; \{\Phi\} = \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{Bmatrix}; \{U\} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

The nodal coordinates are shown in Fig.10.

The variation of the displacement functions for the beam can be written as:

$$\begin{bmatrix} \delta v \\ \delta w \\ \delta \phi \\ \delta u \end{bmatrix} = \begin{bmatrix} \{\Phi_v\}^T & 0 & 0 & 0 \\ 0 & \{\Phi_w\}^T & 0 & 0 \\ 0 & 0 & \{\Phi_\phi\}^T & 0 \\ 0 & 0 & 0 & \{\Phi_u\}^T \end{bmatrix} \begin{bmatrix} \{\delta V\} \\ \{\delta W\} \\ \{\delta \Phi\} \\ \{\delta U\} \end{bmatrix} \tag{6.3}$$

In this development, Hermite interpolation polynomials are used for the space dependent interpolation functions. A cubic Hermite interpolation polynomial, $\{\Phi_c\}$,

is used for the bending deflections (v,w) and a quadratic Hermite interpolation polynomial, $\{\Phi_q\}$, is used for the torsional rotation (ϕ), and the axial deflection (u). The mathematical expressions for these polynomials are given as:

$$\{\Phi_v\} = \{\Phi_w\} = \begin{bmatrix} 1 - 3\xi^2 + 2\xi^3 \\ l_e(\xi - 2\xi^2 + \xi^3) \\ 3\xi^2 - 2\xi^3 \\ l_e(-\xi^2 + \xi^3) \end{bmatrix} = \{\Phi_c\} \tag{6.4}$$

$$\{\Phi_\phi\} = \{\Phi_u\} = \begin{bmatrix} 1 - 3\xi + 2\xi^2 \\ 4\xi - 4\xi^2 \\ -\xi + 2\xi^2 \end{bmatrix} = \{\Phi_q\} \tag{6.5}$$

where:

$\xi = \frac{x}{l_e}$
x = spanwise (axial) coordinate of the beam element.
l_e = length of beam element.

For bending deformations, the nodal parameters are the displacements and slopes at both ends of the beam element. Therefore, the resulting elements will have interelement continuity for both displacements and slopes. In addition, because of the cubic interpolation polynomial, the bending strains vary linearly over the element length.

The quadratic Hermite interpolation polynomial is used for the torsional rotation (ϕ) and the axial deflection (u). This polynomial has the capability of modelling a linear variation of strains along the element length and therefore provides the same level of accuracy as the beam bending element. The nodal parameters for these elements are chosen as the values of the displacement function at the two end nodes and at the mid-point of the element.

The resulting beam element has 14 degrees of freedom: 4 in-plane (lag) bending degrees of freedom, 4 out-plane (flap) degrees of freedom, and 3 degrees of freedom

each for torsion (ϕ), and axial deflection (u). The nodal degrees of freedom are shown in Fig.10.

6.2 ELEMENT MATRICES ASSOCIATED WITH KINETIC ENERGY VARIATION

The beam element matrices associated with the kinetic energy variation are derived by substituting the assumed expressions for the displacement functions (Equation 6.2) in the kinetic energy variation δT (Equation 5.5) and carrying out the integration over the length of the beam element. The resulting variation of the kinetic energy can be written in the form:

$$\begin{aligned} \delta T_i &= -\{\delta q\}^T \Big([M]_{14\times 14}\{\ddot{q}\} + \left[M^C\right]_{14\times 14}\{\dot{q}\} + \left[K^{cf}\right]_{14\times 14}\{q\} + \left\{V^L\right\}_{14\times 1} \\ &\quad + \left[M^1\right]_{14\times 3}\left\{\begin{array}{c}\ddot{R}_x\\ \ddot{R}_y\\ \ddot{R}_z\end{array}\right\} + \left[M^2\right]_{14\times 3}\left\{\begin{array}{c}\dot{R}_x\\ \dot{R}_y\\ \dot{R}_z\end{array}\right\} + \left[M^3\right]_{14\times 3}\left\{\begin{array}{c}\ddot{\theta}_x\\ \ddot{\theta}_y\\ \ddot{\theta}_z\end{array}\right\} + \left[M^4\right]_{14\times 3}\left\{\begin{array}{c}\dot{\theta}_x\\ \dot{\theta}_y\\ \dot{\theta}_z\end{array}\right\} \\ &\quad + \left\{V^I\right\}_{14\times 1} + \left\{V^{NL}\right\}_{14\times 1}\Big) \end{aligned} \tag{6.6}$$

Where {q} represents the vector of unknown nodal degrees of freedom

$$\{q\}_{14\times 1} = \left\{ \begin{array}{c} \{V\} \\ \{W\} \\ \{\Phi\} \\ \{U\} \end{array} \right\} \tag{6.7}$$

Detailed expressions for the various matrices defined in Equation 6.6 are given as follows:

6.2.1 Mass Matrix $[M]_{14\times 14}$

$$[M_{11}] = \int_0^{l_e} \Big[m\{\Phi_c\}\{\Phi_c\}^T + \{Im_{\zeta\zeta}\cos^2 < \theta_G+\phi_k > + Im_{\eta\eta}\sin^2 < \theta_G+\phi_k > \\ -2Im_{\eta\zeta}\sin(\theta_G+\phi_k)\cos(\theta_G+\phi_k)\}\left\{\Phi_c^{'}\right\}\left\{\Phi_c^{'}\right\}^T\Big]\,dx$$

$$[M_{12}] = \int_0^{l_e} \Big[\{(Im_{\zeta\zeta}-Im_{\eta\eta})\sin(\theta_G+\phi_k)\cos(\theta_G+\phi_k) + Im_{\eta\zeta}(\cos^2 < \theta_G+\phi_k > \\ -\sin^2 < \theta_G+\phi_k >)\}\left\{\Phi_c^{'}\right\}\left\{\Phi_c^{'}\right\}^T\Big]\,dx$$

$$[M_{13}] = \int_0^{l_e} \Big[\{-m\eta_m\sin(\theta_G+\phi_k) - m\zeta_m\cos(\theta_G+\phi_k)\}\{\Phi_c\}\{\Phi_q\}^T\Big]\,dx$$

$$[M_{14}] = \int_0^{l_e} \Big[\{-m\eta_m\cos(\theta_G+\phi_k) + m\zeta_m\sin(\theta_G+\phi_k)\}\left\{\Phi_c^{'}\right\}\{\Phi_q\}^T\Big]\,dx$$

$$[M_{21}] = \int_0^{l_e} [\{(Im_{\zeta\zeta}-Im_{\eta\eta})\sin(\theta_G+\phi_k)\cos(\theta_G+\phi_k) \\ +Im_{\eta\zeta}(\cos^2 < \theta_G+\phi_k > -\sin^2 < \theta_G+\phi_k >)\}\left\{\Phi_c^{'}\right\}\left\{\Phi_c^{'}\right\}^T\Big]\,dx$$

$$[M_{22}] = \int_0^{l_e} \Big[m\{\Phi_c\}\{\Phi_c\}^T + \{Im_{\zeta\zeta}\sin^2 < \theta_G+\phi_k > + Im_{\eta\eta}\cos^2 < \theta_G+\phi_k > \\ +2Im_{\eta\zeta}\sin(\theta_G+\phi_k)\cos(\theta_G+\phi_k)\}\left\{\Phi_c^{'}\right\}\left\{\Phi_c^{'}\right\}^T\Big]\,dx$$

$$[M_{23}] = \int_0^{l_e} \Big[\{m\eta_m\cos(\theta_G+\phi_k) - m\zeta_m\sin(\theta_G+\phi_k)\}\{\Phi_c\}\{\Phi_q\}^T\Big]\,dx$$

$$[M_{24}] = \int_0^{l_e} \Big[-\{m\eta_m\sin(\theta_G+\phi_k) + m\zeta_m\cos(\theta_G+\phi_k)\}\left\{\Phi_c^{'}\right\}\{\Phi_q\}^T\Big]\,dx$$

$$[M_{31}] = [M_{13}]^T$$

$$[M_{32}] = [M_{23}]^T$$

$$[M_{33}] = \int_0^{l_e} \left[\{ Im_{\zeta\zeta} + Im_{\eta\eta} \} \{\Phi_q\} \{\Phi_q\}^T \right] dx$$

$$[M_{34}] = [0]$$

$$[M_{41}] = [M_{14}]^T$$

$$[M_{42}] = [M_{24}]^T$$

$$[M_{43}] = [0]$$

$$[M_{44}] = \int_0^{l_e} \left[m \{\Phi_q\} \{\Phi_q\}^T \right] dx$$

6.2.2 Matrix $\left[M^C\right]_{14\times 14}$

$$[M^c_{11}] = \int_0^{l_e} \left[\{-m\eta_m\cos(\theta_G+\phi_k)+m\zeta_m\sin(\theta_G+\phi_k)\}\left\{\Phi_c'\right\}\{\Phi_c\}^T\right]dx$$

$$[M^c_{12}] = \int_0^{l_e} \left[\{m(\beta_p+\beta_d\cos\theta_I+\beta_s\sin\theta_I)\}\{\Phi_c\}\{\Phi_c\}^T\right]dx$$

$$\begin{aligned}[M^c_{13}] = \int_0^{l_e} &\left[-2Im_{\eta\zeta}\sin^2<\theta_G+\phi_k>\left\{\Phi_c'\right\}\{\Phi_q\}^T\right.\\ &\left.+2(Im_{\zeta\zeta}+Im_{\eta\eta})\cos\theta_I\sin(\theta_G+\phi_k)\cos(\theta_G+\phi_k)\left\{\Phi_c'\right\}\{\Phi_q\}^T\right]dx\end{aligned}$$

$$[M^c_{14}] = [0]$$

$$\begin{aligned}[M^c_{21}] = \int_0^{l_e} &[\{m\eta_m\cos<\theta_G+\phi_k> - m\zeta_m\sin<\theta_G+\phi_k> - m\eta_m\sin<\theta_G+\phi_k>\\ &-m\zeta_m\cos<\theta_G+\phi_k>\}(\beta_s\cos\theta_I-\beta_d\sin\theta_I+1)\{\Phi_c\}\left\{\Phi_c'\right\}^T\\ &-\{m\eta_m\sin(\theta_G+\phi_k)+m\zeta_m\cos(\theta_G+\phi_k)\}\left\{\Phi_c'\right\}\{\Phi_c\}^T\\ &\left.+\{m\eta_m\sin(\theta_G+\phi_k)-m\zeta_m\cos(\theta_G+\phi_k)\}\sin\theta_I\{\Phi_c\}\left\{\Phi_c'\right\}^T\right]dx\end{aligned}$$

$$\begin{aligned}[M^c_{22}] &= \int_0^{l_e} [\{m\eta_m\sin(\theta_G+\phi_k)+m\zeta_m\cos(\theta_G+\phi_k)\}(\beta_s\cos\theta_I\\ &= \left.-\beta_d\sin\theta_I+1)\sin\theta_I\{\Phi_c\}\{\Phi_c\}^T\right]dx\end{aligned}$$

$$\begin{aligned}[M^c_{23}] = \int_0^{l_e} &\left[\{-2m\eta_m\sin(\theta_G+\phi_k)(\beta_p+\beta_d\cos\theta_I+\beta_s\sin\theta_I)\}\{\Phi_c\}\{\Phi_q\}^T\right.\\ &-\{m\eta_m\sin(\theta_G+\phi_k)+m\zeta_m\cos(\theta_G+\phi_k)\}\{\Phi_c\}\{\Phi_q\}^T\\ &-a\{m\eta_m\cos(\theta_G+\phi_k)-m\zeta_m\sin(\theta_G+\phi_k)\}\left\{\Phi_c'\right\}\{\Phi_q\}^T\\ &-2Im_{\eta\eta}\sin^2<\theta_G+\phi_k>\left\{\Phi_c'\right\}\{\Phi_q\}^T\\ &\left.+\{(Im_{\zeta\zeta}-Im_{\eta\eta})\sin\theta_I-2Im_{\eta\zeta}\}\sin(\theta_G+\phi_k)\cos(\theta_G+\phi_k)\left\{\Phi_c'\right\}\{\Phi_q\}^T\right]dx\end{aligned}$$

$$[M^c_{24}] = [0]$$

$$
\begin{aligned}
[M^c_{31}] &= \int_0^{l_e} \Big[\{(Im_{\zeta\zeta} + Im_{\eta\eta})\sin(\theta_G + \phi_k)\cos(\theta_G + \phi_k) - 2Im_{\eta\zeta}\cos^2(\theta_G + \phi_k) \\
&\quad +2(Im_{\zeta\zeta} + Im_{\eta\eta})\sin(\theta_G + \phi_k)\cos(\theta_G + \phi_k)\sin\theta_I \\
&\quad -(Im_{\zeta\zeta} - Im_{\eta\eta})\sin(\theta_G + \phi_k)\cos(\theta_G + \phi_k) \\
&\quad -2Im_{\eta\zeta}\cos^2 < \theta_G + \phi_k >\}\cos\theta_I \{\Phi_q\}\left\{\Phi_c'\right\}^T\Big]\, dx
\end{aligned}
$$

$$
\begin{aligned}
[M^c_{32}] &= \int_0^{l_e} \Big[\{-Im_{\zeta\zeta}\cos^2(\theta_G + \phi_k) - Im_{\eta\eta} + Im_{\eta\eta}\sin(\theta_G + \phi_k)\cos(\theta_G + \phi_k) \\
&\quad -Im_{\eta\zeta}(2\sin < \theta_G + \phi_k > \cos < \theta_G + \phi_k >) \\
&\quad +(Im_{\zeta\zeta} - Im_{\eta\eta})\cos^2 < \theta_G + \phi_k > \\
&\quad +2(Im_{\zeta\zeta} + Im_{\eta\eta})\sin(\theta_G + \phi_k)\cos(\theta_G + \phi_k)\sin\theta_I \\
&\quad +m\eta_m\sin(\theta_G + \phi_k)(\beta_p + \beta_d\cos\theta_I + \beta_s\sin\theta_I)\}\{\Phi_q\}\left\{\Phi_c'\right\}^T\Big]\, dx
\end{aligned}
$$

$$
\begin{aligned}
[M^c_{33}] &= \int_0^{l_e} [[\{m\eta_m\cos(\theta_G + \phi_k) - m\zeta_m\sin(\theta_G + \phi_k)\}(\beta_d\cos\theta_I + \beta_s\sin\theta_I)(< n-1 > l_e + x_k) \\
&\quad -\{m\eta_m\sin(\theta_G + \phi_k) - m\zeta_m\cos(\theta_G + \phi_k)\} \\
&\quad \{a - (\beta_s\cos\theta_I - \beta_d\sin\theta_I + 1)(< n-1 > l_e + x_k) \\
&\quad -(e_1 + e_2)\} - (Im_{\zeta\zeta} + Im_{\eta\eta})\sin(\theta_G + \phi_k)\cos(\theta_G + \phi_k)(\beta_p + \beta_d\cos\theta_I + \beta_s\sin\theta_I) \\
&\quad -(Im_{\zeta\zeta} + Im_{\eta\eta})\sin(\theta_G + \phi_k)\cos(\theta_G + \phi_k)(\beta_p + \beta_d\cos\theta_I + \beta_s\sin\theta_I) \\
&\quad -Im_{\eta\zeta}\{3\sin^2(\theta_G + \phi_k) - 5\cos^2(\theta_G + \phi_k)\}(\beta_p + \beta_d\cos\theta_I + \beta_s\sin\theta_I)\{\Phi_q\}\{\Phi_q\}^T\Big]\, dx
\end{aligned}
$$

$$
\begin{aligned}
[M^c_{34}] &= \int_0^{l_e} \big[-m\cos\theta_I + \{m\eta_m\cos(\theta_G + \phi_k) - m\zeta_m\sin(\theta_G + \phi_k)\sin\theta_I \\
&\quad -\{m\eta_m\sin(\theta_G + \phi_k) - m\zeta_m\cos(\theta_G + \phi_k)\}\cos\theta_I\{\Phi_q\}\{\Phi_q\}^T\Big]\, dx
\end{aligned}
$$

$$
[M^c_{41}] = \int_0^{l_e} \left[m\cos\theta_I \{\Phi_q\}\{\Phi_c\}^T\right] dx
$$

$$
[M^c_{42}] = \int_0^{l_e} \left[m\sin\theta_I \{\Phi_q\}\{\Phi_c\}^T\right] dx
$$

$$[M_{43}^c] = \int_0^{l_e} \left[\{-2m\dot{\eta}_m \sin(\theta_G + \phi_k)\} \{\Phi_q\} \{\Phi_q\}^T \right] dx$$

$$\begin{aligned}[M_{44}^c] \quad & \int_0^{l_e} \left[\{m\eta_m \sin(\theta_G + \phi_k) - m\zeta_m \cos(\theta_G + \phi_k)\} \cos\theta_I \{\Phi_q\} \{\Phi_q\}^T \right. \\ & \left. -\{(m\eta_m \cos(\theta_G + \phi_k) - m\zeta_m \cos(\theta_G + \phi_k)\} \sin\theta_I \{\Phi_q\} \{\Phi_q\}^T \right] dx\end{aligned}$$

6.2.3 Matrix $\left[K^{CF}\right]_{14\times 14}$

$$\begin{aligned}\left[K_{11}^{cf}\right] &= \int_0^{l_e}\Big[-m\cos\theta_I\left\{\Phi_c\right\}\left\{\Phi_c\right\}^T \\ &+\{Im_{\zeta\zeta}\cos^2<\theta_G+\phi_k> +Im_{\eta\eta}\sin^2<\theta_G+\phi_k>\}\cos\theta_I\left\{\Phi_c^{'}\right\}\left\{\Phi_c^{'}\right\}^T \\ &-2Im_{\eta\zeta}\sin(\theta_G+\phi_k)\cos(\theta_G+\phi_k)\cos\theta_I\left\{\Phi_c^{'}\right\}\left\{\Phi_c^{'}\right\}^T\Big]\,dx\end{aligned}$$

$$\begin{aligned}\left[K_{12}^{cf}\right] &= \int_0^{l_e}\big[[-(Im_{\zeta\zeta}-Im_{\eta\eta})\sin(\theta_G+\phi_k)\cos(\theta_G+\phi_k) \\ &-Im_{\eta\zeta}\{\cos^2(\theta_G+\phi_k)-\sin^2(\theta_G+\phi_k)\} \\ &+m\eta_m\cos(\theta_G+\phi_k)-m\zeta_m\sin(\theta_G+\phi_k)]\cos\theta_I\left\{\Phi_c^{'}\right\}\left\{\Phi_c^{'}\right\}^T \\ &+m\sin\theta_I\cos\theta_I\left\{\Phi_c\right\}\left\{\Phi_c\right\}^T\big]\,dx\end{aligned}$$

$$\left[K_{13}^{cf}\right] = [0]$$

$$\left[K_{14}^{cf}\right] = [0]$$

$$\begin{aligned}\left[K_{21}^{cf}\right] &= \int_0^{l_e}\big[[(Im_{\zeta\zeta}-Im_{\eta\eta})\sin(\theta_G+\phi_k)\cos(\theta_G+\phi_k) \\ &+Im_{\eta\zeta}\{\sin^2(\theta_G+\phi_k)-\cos^2(\theta_G+\phi_k)\}]\cos\theta_I\left\{\Phi_c^{'}\right\}\left\{\Phi_c^{'}\right\}^T \\ &+m\sin\theta_I\left\{\Phi_c\right\}\left\{\Phi_c\right\}^T\big]\,dx\end{aligned}$$

$$\begin{aligned}\left[K_{22}^{cf}\right] &= \int_0^{l_e}\big[-\{m\eta_m\cos(\theta_G+\phi_k)-m\zeta_m\sin(\theta_G+\phi_k)\} \\ &(\beta_s\cos\theta_I-\beta_d\sin\theta_I+1)\left\{\Phi_c\right\}\left\{\Phi_c^{'}\right\}^T \\ &-\{Im_{\zeta\zeta}\sin^2(\theta_G+\phi_k)+Im_{\eta\eta}\cos^2(\theta_G+\phi_k) \\ &+2Im_{\eta\zeta}\sin(\theta_G+\phi_k)\cos(\theta_G+\phi_k)\}\left\{\Phi_c^{'}\right\}\left\{\Phi_c^{'}\right\}^T \\ &+m\sin\theta_I\left\{\Phi_c\right\}\left\{\Phi_c\right\}^T\big]\,dx\end{aligned}$$

$$\left[K^{cf}_{23}\right] = [0]$$
$$\left[K^{cf}_{24}\right] = [0]$$

$$\begin{aligned}\left[K^{cf}_{31}\right] = &\int_0^{l_e} [[\{m\eta_m \sin(\theta_G+\phi_k) - m\zeta_m \cos(\theta_G+\phi_k)\} + \{Im_{\zeta\zeta}\sin(\theta_G+\phi_k)\\ &-Im_{\eta\eta}\cos(\theta_G+\phi_k)\}(\beta_p+\beta_d\cos\theta_I+\beta_s\sin\theta_I)]\cos\theta_I\,\{\Phi_q\}\,\{\Phi_c\}^T\\ &+[(Im_{\zeta\zeta}+Im_{\eta\eta})(\beta_p+\beta_d\cos\theta_I+\beta_s\sin\theta_I)\\ &-\{m\eta_m\sin(\theta_G+\phi_k)+m\zeta_m\cos(\theta_G+\phi_k)\}\\ &(e_1+e_2+<n-1>l_e+x_k)\,\{\Phi_q\}\left\{\Phi^{'}_c\right\}^T\Big]\,dx\end{aligned}$$

$$\begin{aligned}\left[K^{cf}_{32}\right] = &\int_0^{l_e} [[(Im_{\eta\eta}-Im_{\zeta\zeta})\sin(\theta_G+\phi_k)\cos(\theta_G+\phi_k)(\beta_p+\beta_d\cos\theta_I+\beta_s\sin\theta_I)\\ &+\{m\eta_m\cos(\theta_G+\phi_k)-m\zeta_m\sin(\theta_G+\phi_k)\}(e_1+e_2+<n-1>l_e x_k)\\ &+2Im_{\eta\zeta}\{\sin^2<\theta_G+\phi_k>-\cos^2<\theta_G+\phi_k>\}\\ &(\beta_p+\beta_d\cos\theta_I+\beta_s\sin\theta_I]\cos\theta_I\,\{\Phi_q\}\left\{\Phi^{'}_c\right\}^T\Big]\,dx\end{aligned}$$

$$\left[K^{cf}_{33}\right] = [0]$$
$$\left[K^{cf}_{34}\right] = [0]$$

$$\left[K^{cf}_{41}\right] = \int_0^{l_e}\left[-\{m\eta_m\cos(\theta_G+\phi_k)-m\zeta_m\sin(\theta_G+\phi_k)\}\cos\theta_I\,\{\Phi_q\}\left\{\Phi^{'}_c\right\}^T\right]dx$$

$$\left[K^{cf}_{42}\right] = \int_0^{l_e}\left[m(\beta_p+\beta_d\cos\theta_I+\beta_s\sin\theta_I)\cos\theta_I\,\{\Phi_q\}\left\{\Phi^{'}_c\right\}^T\right]dx$$

$$\left[K^{cf}_{43}\right] = [0]$$

$$\left[K^{cf}_{44}\right] = \int_0^{l_e}\left[-m(\cos\theta_I+\sin^2\theta_I\,\{\Phi_q\}\,\{\Phi_q\}^T\right]dx$$

6.2.4 Vector $\left[V^L\right]_{14\times 1}$

$$\begin{aligned}
\left[V_{11}^L\right] &= \int_0^{l_e} \Big[-\{m\eta_m \sin(\theta_G+\phi_k) + m\zeta_m \cos(\theta_G+\phi_k)\}\cos^2\theta_I\,\{\Phi_c\} \\
&\quad \{ma(2\cos\theta_I - \sin\theta_I)\cos\theta_I - m(e_1+e_2)(\beta_s\cos\theta_I - \beta_d\sin\theta_I)\cos^2\theta_I\}\,\{\Phi_c\} \\
&\quad -\{m\eta_m \sin(\theta_G+\phi_k) + m\zeta_m\cos(\theta_G+\phi_k)\}\,\{\Phi_c\} \\
&\quad +\{m\eta_m\cos(\theta_G+\phi_k) - m\zeta_m\sin(\theta_G+\phi_k)\}(< n-1 > l_e + x_k)\left\{\Phi_c^{'}\right\}\Big]\,dx
\end{aligned}$$

$$\begin{aligned}
\left[V_{21}^L\right] &= \int_0^{l_e} [\{-m(e_1+e_2+ < n-1 > l_e + x_k)(\beta_p + \beta_d\cos\theta_I + \beta_s\sin\theta_I)\}\,\{\Phi_c\} \\
&\quad \{ma(2\cos\theta_I - \sin\theta_I)\sin\theta_I\,\{\Phi_c\} \\
&\quad +\{m\eta_m\sin(\theta_G+\phi_k) + m\zeta_m\cos(\theta_G+\phi_k)\}(< n-1 > l_e + x_k)\left\{\Phi_c^{'}\right\}\Big]\,dx
\end{aligned}$$

$$\begin{aligned}
\left[V_{31}^L\right] &= \int_0^{l_e} \Big[[\{(Im_{\zeta\zeta} - Im_{\eta\eta})\sin(\theta_G+\phi_k)\cos(\theta_G+\phi_k)\}\cos^2\theta_I \\
&\quad -Im_{\eta\zeta}\{\sin^2(\theta_G+\phi_k) - \cos^2(\theta_G+\phi_k)\}\cos^2\theta_I \\
&\quad +\{Im_{\zeta\zeta}\sin(\theta_G+\phi_k) + Im_{\eta\eta}\cos(\theta_G+\phi_k)\}\sin\theta_I]\,\{\Phi_q\}]\,dx
\end{aligned}$$

$$\begin{aligned}
\left[V_{41}^L\right] &= \int_0^{l_e} [-m\{e_1+e_2+ < n-1 > l_e + x_k + (\beta_d\cos\theta_I + \beta_s\sin\theta_I)(< n-1 > l_e + x_k) \\
&\quad -a(\beta_s\cos\theta_I - \beta_d\sin\theta_I)\}\cos\theta_I\,\{\Phi_q\}]\,dx
\end{aligned}$$

Using small angle approximation for ϕ_k, the vector $\left[V_{31}^L\right]$ can be written in a modified form, which is given in Appendix B.

6.2.5 Matrix $\left[M^1\right]_{14\times 3}$

$$\left[M^1_{11}\right] = \int_0^{l_e} \left[\{-m\sin\psi_k\}\cos\theta_I\left\{\Phi_c\right\}\right]dx$$

$$\left[M^1_{12}\right] = \int_0^{l_e} \left[\{m\cos\psi k\}\cos\theta_I\left\{\Phi_c\right\}\right]dx$$

$$\left[M^1_{13}\right] = \int_0^{l_e} \left[-m\sin\theta_I\left\{\Phi_c\right\}\right]dx$$

$$\left[M^1_{21}\right] = \int_0^{l_e} \left[\{-m\cos\psi k\}\left\{\Phi_c\right\}\right]dx$$

$$\left[M^1_{22}\right] = \int_0^{l_e} \left[\{-m\sin\psi_k\}\left\{\Phi_c\right\}\right]dx$$

$$\left[M^1_{23}\right] = \int_0^{l_e} \left[m\left\{\Phi_c\right\}\right]dx$$

$$\left[M^1_{31}\right] = [0]$$

$$\left[M^1_{32}\right] = [0]$$

$$\left[M^1_{33}\right] = [0]$$

$$\left[M^1_{41}\right] = \int_0^{l_e} \left[\{-m\cos\psi k\}\left\{\Phi_q\right\}\right]dx$$

$$\left[M^1_{42}\right] = \int_0^{l_e} \left[\{m\sin\psi_k\}\left\{\Phi_q\right\}\right]dx$$

$$\left[M^1_{43}\right] = \int_0^{l_e} \left[m(\beta_p+\beta_d\cos\theta_I+\beta_s\sin\theta_I)\left\{\Phi_q\right\}\right]dx$$

6.2.6 Matrix $\left[M^2\right]_{14\times3}$

$$\left[M^2_{11}\right] = \int_0^{l_e} \left[-m\{2\cos\psi k + (\beta_s\cos\theta_I - \beta_d\sin\theta_I)\sin\psi_k\}\cos\theta_I\,\{\Phi_c\}\right] dx$$

$$\left[M^2_{12}\right] = \int_0^{l_e} \left[-m\{2\sin\psi_k + (-\beta_s\cos\theta_I - \beta_d\sin\theta_I)\cos\psi k\}\cos\theta_I\,\{\Phi_c\}\right] dx$$

$$\left[M^2_{13}\right] = \int_0^{l_e} \left[m\dot{\theta}_I\,\{\Phi_c\}\right] dx$$

$$\left[M^2_{21}\right] = \int_0^{l_e} \left[m\{\sin\psi_k(\beta_p + \beta_d\cos\theta_I + \beta_s\sin\theta_I - \dot{\theta}_I) - \dot{\theta}_I\cos\psi k\}\,\{\Phi_c\}\right] dx$$

$$\left[M^2_{22}\right] = \int_0^{l_e} \left[m\{\cos\psi k(\beta_p + \beta_d\cos\theta_I + \beta_s\sin\theta_I - \dot{\theta}_I) - \dot{\theta}_I\sin\psi_k\}\,\{\Phi_c\}\right] dx$$

$$\left[M^2_{23}\right] = [0]$$

$$\left[M^2_{31}\right] = [0]$$

$$\left[M^2_{32}\right] = [0]$$

$$\left[M^2_{33}\right] = [0]$$

$$\left[M^2_{41}\right] = \int_0^{l_e} \left[m\cos\psi k(\beta_s\cos\theta_I - \beta_d\sin\theta_I)\cos\theta_I\,\{\Phi_q\}\right] dx$$

$$\left[M^2_{42}\right] = \int_0^{l_e} \left[m\{2\cos\psi k + (\beta_s\cos\theta_I - \beta_d\sin\theta_I)\sin\psi_k\}\cos\theta_I\,\{\Phi_q\}\right] dx$$

$$\left[M^2_{43}\right] = [0]$$

6.2.7 Matrix $\left[M^3\right]_{14\times 3}$

$$\left[M^3_{11}\right] = [0]$$

$$\left[M^3_{12}\right] = [0]$$

$$\left[M^3_{13}\right] = [0]$$

$$\left[M^3_{21}\right] = \int_0^{l_e} \left[m\{a\cos\psi k + (e_1+e_2)\sin\psi_k\}\cos\theta_I\{\Phi_c\}\right] dx$$

$$\left[M^3_{21}\right] = \int_0^{l_e} \left[m\{a\sin\psi_k - (e_1+e_2)\cos\psi k\}\cos\theta_I\{\Phi_c\}\right] dx$$

$$\left[M^3_{23}\right] = [0]$$

$$\left[M^3_{31}\right] = [0]$$

$$\left[M^3_{32}\right] = [0]$$

$$\left[M^3_{33}\right] = [0]$$

$$\left[M^3_{41}\right] = [0]$$

$$\left[M^3_{42}\right] = [0]$$

$$\left[M^3_{43}\right] = \int_0^{l_e} \left[-ma\cos\theta_I\{\Phi_q\}\right] dx$$

6.2.8 Matrix $\left[M^4\right]_{14\times 3}$

$$\begin{aligned}
\left[M^4_{11}\right] &= \int_0^{l_e} [[-m\{a\cos\theta_I - (\beta_s\cos\theta_I - \beta_d\sin\theta_I + 1)(< n-1 > l_e + x_k) \\
&\quad -(e_1+e_2)\}\cos\psi k + m\{(< n-1 > l_e + x_k)\sin\psi_k\}]\{\Phi_c\}]\,dx
\end{aligned}$$

$$\begin{aligned}
\left[M^4_{12}\right] &= \int_0^{l_e} [[-m\{a\cos\theta_I - (\beta_s\cos\theta_I - \beta_d\sin\theta_I + 1)(< n-1 > l_e + x_k) \\
&\quad -(e_1+e_2)\}\sin\psi_k - m\{(< n-1 > l_e + x_k)\cos\psi k\}]\{\Phi_c\}]\,dx
\end{aligned}$$

$$\left[M^4_{13}\right] = \int_0^{l_e} [ma\cos\theta_I(2\cos\theta_I - \sin\theta_I)\{\Phi_c\}\{\Phi_c\}]\,dx$$

$$\left[M^4_{21}\right] = \int_0^{l_e} [-m\{2a\sin\psi_k\cos\theta_I + (< n-1 > l_e + x_k)\cos\psi k\}\{\Phi_c\}]\,dx$$

$$\left[M^4_{22}\right] = \int_0^{l_e} [m\{2a\cos\psi k\cos\theta_I - (< n-1 > l_e + x_k)\sin\psi_k\}\{\Phi_c\}]\,dx$$

$$\left[M^4_{23}\right] = \int_0^{l_e} \left[\{-m(< n-1 > l_e + x_k)(\beta_p + \beta_d\cos\theta_I + \beta_s\sin\theta_I - \dot{\theta}_I)\}\{\Phi_c\}\right]dx$$

$$\begin{aligned}
\left[M^4_{31}\right] &= [0] \\
\left[M^4_{32}\right] &= [0] \\
\left[M^4_{33}\right] &= [0] \\
\left[M^4_{41}\right] &= \int_0^{l_e} [-m\{a\cos\theta_I - (\beta_s\cos\theta_I - \beta_d\sin\theta_I + 1)(< n-1 > l_e + x_k) \\
&\quad -(e_1+e_2)\}\sin\psi_k\{\Phi_q\}]\,dx
\end{aligned}$$

$$\begin{aligned}
\left[M^4_{42}\right] &= \int_0^{l_e} [m\{a\cos\theta_I - (\beta_s\cos\theta_I - \beta_d\sin\theta_I + 1)(< n-1 > l_e + x_k) \\
&\quad -(e_1+e_2)\}\cos\psi k\{\Phi_q\}]\,dx
\end{aligned}$$

$$\left[M^4_{43}\right] = \int_0^{l_e} [m\{(e_1+e_2) + 2(< n-1 > l_e + x_k)\}\cos\theta_I\{\Phi_q\}]\,dx$$

6.2.9 Vector $\left[V^I\right]_{14\times 1}$

$$\begin{aligned}
\left[V^I_{11}\right] &= \int_0^{l_e} \left[-m[\{3a-(e_1+e_2)-(\beta_s\cos\theta_I-\beta_d\sin\theta_I+1)(<n-1>l_e+x_k)\right.\\
&\quad -(\beta_d<\cos\theta_I+\sin\theta_I>+\beta_s<\sin\theta_I-\cos\theta_I>)\}\dot{\theta}_I\\
&\quad -\{(\beta_d\cos\theta_I+\beta_s\sin\theta_I-1)(<n-1>l_e+x_k)\}\ddot{\theta}_I\\
&\quad \left.+\{a+(\beta_d\sin\theta_I+\beta_s\cos\theta_I)(<n-1>l_e+x_k)\}\dot{\theta}_I^2]\,\{\Phi_c\}\right] dx
\end{aligned}$$

$$\begin{aligned}
\left[V^I_{21}\right] &= \int_0^{l_e} \left[-m[\{a+(\beta_d\cos\theta_I+\beta_s\sin\theta_I)(<n-1>l_e+x_k)\}\dot{\theta}_I\right.\\
&\quad -\{a\cos\theta_I-(\beta_s\cos\theta_I-\beta_d\sin\theta_I+1)(<n-1>l_e+x_k)-(e_1+e_2)\}\ddot{\theta}_I\\
&\quad \left.-(\beta_s\sin\theta_I+\beta_d\cos\theta_I)(<n-1>l_e+x_k)\dot{\theta}_I^2]\,\{\Phi_c\}\right] dx
\end{aligned}$$

$$\left[V^I_{31}\right] = [0]$$

$$\begin{aligned}
\left[V^I_{41}\right] &= \int_0^{l_e} \left[-m[\{2(<n-1>l_e+x_k)(\beta_d\cos\theta_I+\beta_s\sin\theta_I)\}\dot{\theta}_I+a\ddot{\theta}_I\right.\\
&\quad -\{(e_1+e_2+<n-1>l_e+x_k)(1+\beta_d\cos\theta_I+\beta_s\sin\theta_I)\\
&\quad \left.-(e_1+e_2-a)\}\dot{\theta}_I^2+a\sin\theta_I\cos\theta_I\dot{\theta}_I]\,\{\Phi_q\}\right] dx
\end{aligned}$$

6.2.10 Nonlinear Vector $\left[V^{NL}\right]_{14\times 1}$

$$
\begin{aligned}
\left[V_{11}^{NL}\right] &= \int_0^{l_e} \Big[-m[2\dot{\theta}_I \cos\theta_I \{\Phi_c\}\{\Phi_c\}^T \{V\} + (\dot{\theta}_I^2 \cos\theta_I + \dot{\theta}_I\dot{\theta}_z - \ddot{\theta}_I)\{\Phi_c\}\{\Phi_c\}^T \{W\} \\
&\quad + \ddot{\theta}_I \{\Phi_c\}\{\Phi_q\}^T \{U\} + 2\dot{\theta}_I \{\Phi_c\}\{\Phi_q\}^T \left\{\dot{U}\right\}]\Big]\, dx
\end{aligned}
$$

$$
\begin{aligned}
\left[V_{21}^{NL}\right] &= \int_0^{l_e} \Big[-m[(\dot{\theta}_I - \ddot{\theta}_I)\{\Phi_c\}\{\Phi_c\}^T \{V\} - \dot{\theta}_I \{\Phi_c\}\{\Phi_c\}^T \left\{\dot{V}\right\} \\
&\quad + \dot{\theta}_I^2 \{\Phi_c\}\{\Phi_c\}^T \{W\}]\Big]\, dx
\end{aligned}
$$

$$
\left[V_{31}^{NL}\right] = [0]
$$

$$
\begin{aligned}
\left[V_{41}^{NL}\right] &= \int_0^{l_e} \Big[m[(-\dot{\theta}_I)\{\Phi_q\}\{\Phi_c\}^T \left\{\dot{V}\right\} - (\dot{\theta}_I + \dot{\theta}_I^2)\cos\theta_I \{\Phi_q\}\{\Phi_c\}^T \{W\} \\
&\quad - \dot{\theta}_I \{\Phi_q\}\{\Phi_q\}^T \{U\}]\Big]\, dx
\end{aligned}
$$

6.3 ELEMENT MATRICES ASSOCIATED WITH STRAIN ENERGY VARIATION

The element matrices associated with the strain energy variation are derived by substituting the assumed expressions for the displacement functions (Equation 6.2), in the strain energy variation, δU (Equation 5.9), and carrying out the integration over the length of the beam element. The resulting variation of the strain energy can be written in the form:

$$\delta U = E_o l^3 \{\delta q\}^T \left\{ \left[K^E\right] \{q\} + \left\{F^E\right\} \right\} \tag{6.8}$$

Where

{q} represents the vector of unknown degree of freedom.

$\left[K^E\right]$ is the linear stiffness matrix. $\left\{F^E\right\}$ is the nonlinear stiffness vector.

The detailed expressions for the $\left[K^E\right]$ and $\left\{F^E\right\}$ are as follows:

6.3.1 Linear Stiffness Matrix $\left[K^E\right]$

The linear stiffness matrix $[K^E]$ is given by the following sub-matrices:

$$\left[K_{11}^E\right] = \int_0^{l_e} \left[(\overline{EI_{\zeta\zeta}} \cos\theta_g - \overline{EI_{\eta\zeta}} \sin\theta_g) \{\Phi_c''\} \{\Phi_c''\}^T \right] dx$$

$$\left[K_{12}^E\right] = \int_0^{l_e} \left[(\overline{EI_{\zeta\eta}} \cos\theta_g - \overline{EI_{\eta\eta}} \sin\theta_g) \{\Phi_c''\} \{\Phi_c''\}^T \right] dx$$

$$\left[K_{13}^E\right] = \int_0^{l_e} \left[(\overline{EAD_1}) \{\Phi_c''\} \left\{\Phi_q''\right\}^T + (\tau_o \overline{EAD_1}') \{\Phi_c''\} \left\{\Phi_q'\right\}^T \right] dx$$

$$\left[K_{14}^E\right] = \int_0^{l_e} \left[(-\overline{EA\eta_a}) \{\Phi_c''\} \left\{\Phi_q'\right\}^T \right] dx$$

$$\left[K_{21}^E\right] = \left[K_{12}^E\right]^T$$

$$\left[K_{22}^{E}\right] = \int_0^{l_e} \left[(\overline{EI_{\eta\eta}}\cos\theta_g - \overline{EI_{\zeta\eta}}\sin\theta_g)\left\{\Phi_c''\right\}\left\{\Phi_c''\right\}^T\right] dx$$

$$\left[K_{23}^{E}\right] = \int_0^{l_e} \left[(\overline{EAD_2})\left\{\Phi_c''\right\}\left\{\Phi_q''\right\}^T + (\tau_o\overline{EAD_2}')\left\{\Phi_c''\right\}\left\{\Phi_q'\right\}^T\right] dx$$

$$\left[K_{24}^{E}\right] = \int_0^{l_e} \left[(-\overline{EA\zeta_a})\left\{\Phi_c''\right\}\left\{\Phi_q'\right\}^T\right] dx$$

$$\left[K_{31}^{E}\right] = \left[K_{13}^{E}\right]^T$$

$$\left[K_{32}^{E}\right] = \left[K_{23}^{E}\right]^T$$

$$\begin{aligned}\left[K_{33}^{E}\right] = & \int_0^{l_e}[(EAD_3)\left\{\Phi_q''\right\}\left\{\Phi_q''\right\}^T + (\tau_o EAD_5)(\left\{\Phi_q''\right\}\left\{\Phi_q'\right\}^T + \left\{\Phi_q'\right\}\left\{\Phi_q''\right\}^T) \\ & +(\tau_o^2 EAD_3' + GJ)\left\{\Phi_q'\right\}\left\{\Phi_q'\right\}^T]dx\end{aligned}$$

$$\left[K_{34}^{E}\right] = \int_0^{l_e} \left[(-EAD_o)\left\{\Phi_q''\right\}\left\{\Phi_q''\right\}^T + (-\tau_o EAD_2')\left\{\Phi_q'\right\}\left\{\Phi_q'\right\}^T\right] dx$$

$$\left[K_{41}^{E}\right] = \left[K_{14}^{E}\right]^T$$

$$\left[K_{42}^{E}\right] = \left[K_{24}^{E}\right]^T$$

$$\left[K_{43}^{E}\right] = \left[K_{34}^{E}\right]^T$$

$$\left[K_{44}^{E}\right] = \int_0^{l_e} \left[(EA)\left\{\Phi_q'\right\}\left\{\Phi_q'\right\}^T\right] dx$$

6.3.2 Simplification of Linear Stiffness Matrix $[K^E]$

A simplified form of the linear stiffness matrix can be obtained by making the following assumptions:

1. The elastic axis is coincident with the modulus weighted centroid.
2. The e_η, e_ς are chosen parallel to the principal axis.
3. Warping function is assumed to be zero.

With these assumptions $EA, EI_{\eta\eta}, EI_{\varsigma\varsigma}, EAC_o$ are the only integrals that remain non-zero. The linear stiffness matrix can be written as:

$$[K^E] = \begin{bmatrix} (EI_{\varsigma\varsigma}\cos^2\theta_g)[A] & (-EI_{\eta\eta}\sin\theta_g\cos\theta_g)[A] & [O] & [O] \\ (-EI_{\eta\eta}\sin\theta_g\cos\theta_g)[A] & (EI_{\eta\eta}\cos^2\theta_g)[A] & [O] & [O] \\ [O] & [O] & (GJ)[B] & [O] \\ [O] & [O] & [O] & (EA)[B] \end{bmatrix} \tag{6.9}$$

where

$$[A] = \frac{1}{l_e^3}\begin{bmatrix} 12 & 6l_e & -12 & 6l_e \\ 6l_e & 4l_e^2 & -6l_e & 2l_e^2 \\ -12 & -6l_e & 12 & -6l_e \\ 6l_e & 2l_e^2 & -6l_e & 4l_e^2 \end{bmatrix}$$

$$[B] = \frac{1}{3l_e}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix}$$

6.3.3 Nonlinear Stiffness Vector $[F^E]$

The nonlinear stiffness vector $[F^E]$ is given by the following sub-vectors:

$$\begin{aligned}\left\{F_1^E\right\} &= \int_0^{l_e}\{[\underline{(\bar{V}_x)(v_x)} - (\bar{S}_x{}'\sin\theta_g)(v_{xx}\cos\theta_g + w_{xx}\sin\theta_g)]\,\{\Phi_c'\}\\ &+[-(\overline{EI_{\zeta\eta}}\cos\theta_g - \overline{EI_{\eta\eta}}\sin\theta_g)(\phi)(v_{xx}) + (\overline{EI_{\zeta\zeta}}\cos\theta_g - \overline{EI_{\eta\zeta}}\sin\theta_g)(\phi)(w_{xx})\\ &-\frac{1}{2}(\overline{EA\eta_a})(v_x^2+w_x^2) + (\bar{S}_x{}'\cos\theta_g)(-v_x\sin\theta_g + w_x\cos\theta_g)\\ &+(\bar{M}_\eta{}'\cos\theta_g - \bar{M}_\zeta{}'\sin\theta_g)(\phi) - \frac{1}{2}(\overline{EAC_1})(\phi_x^2)]\,\{\Phi_c''\}\}dx\end{aligned}$$

$$\begin{aligned}\left\{F_2^E\right\} &= \int_0^{l_e}\{[\underline{(\bar{V}_x)(w_x)} + (\bar{S}_x{}'\cos\theta_g)(v_{xx}\cos\theta_g + w_{xx}\sin\theta_g)]\,\{\Phi_c'\}\\ &+[-(\overline{EI_{\zeta\eta}}\sin\theta_g + \overline{EI_{\eta\eta}}\cos\theta_g)(\phi)(v_{xx}) + (\overline{EI_{\zeta\zeta}}\sin\theta_g + \overline{EI_{\eta\zeta}}\cos\theta_g)(\phi)(w_{xx})\\ &-\frac{1}{2}(\overline{EA\zeta_a})(v_x^2+w_x^2) + (\bar{S}_x{}'\sin\theta_g)(-v_x\sin\theta_g + w_x\cos\theta_g)\\ &+(\bar{M}_\eta{}'\sin\theta_g + \bar{M}_\zeta{}'\cos\theta_g)(\phi) - \frac{1}{2}(\overline{EAC_2})(\phi_x^2)]\,\{\Phi_c''\}\}dx\end{aligned}$$

$$\begin{aligned}\left\{F_3^E\right\} &= \int_0^{l_e}\{[(\bar{M}_\eta{}'\cos\theta_g - \bar{M}_\zeta{}'\sin\theta_g)(v_{xx}) + (\bar{M}_\eta{}'\sin\theta_g + \bar{M}_\zeta{}'\cos\theta_g)(w_{xx})]\{\Phi_q\}\\ &-[(\tau_o)[(\overline{EAD_2'}v_{xx} - \overline{EAD_1'}w_{xx})(\phi) + \frac{1}{2}EAD_o'(v_x^2+w_x^2) + \frac{1}{2}EAD_4'(\phi_x^2)]\\ &-(GJ_o - GJ_1)(\phi_o) - \underline{(\bar{T}_x)(\phi_x)}]\{\Phi_q'\}\\ &-[\frac{1}{2}EAD_o(v_x^2+w_x^2) + \frac{1}{2}EAD_4(\phi_x^2) + (\overline{EAD_2}v_{xx} - \overline{EAD_1}w_{xx})\phi]\{\Phi_q''\}\}dx\end{aligned}$$

$$\left\{F_4^E\right\} = \int_0^{l_e}\{[(\overline{EA\zeta_a}v_{xx} - \overline{EA\eta_a}w_{xx})(\phi) + \frac{1}{2}EA(v_x^2+w_x^2) + \frac{1}{2}EAC_o(\phi_x^2)]\{\Phi_q'\}\}dx$$

The underlined terms in $\left\{F_1^E\right\}$,$\left\{F_2^E\right\}$ and $\left\{F_3^E\right\}$ are associated with the axial strain at the elastic axis $(u_x+\frac{1}{2}v_x^2+\frac{1}{2}w_x^2)$ of the blade. These nonlinear terms are modified to a set of linear terms by substituting the axial stress by the axial inertia force. The discription of the substitution approach [23,26] is given in next section for clarity.

6.3.4 Treatment of Nonlinear Terms Associated with Axial Strain at Elastic Axis

The equation of motion corresponding to the axial degree of freedom can be written symbolically as [23,26] :

$$\frac{E_o}{m\Omega^2}[-\bar{V}_x]_x - \bar{Z}_u = 0 \tag{6.10}$$

The axial stress resultant (Equation 5.10) can be rewritten as:

$$\bar{V}_x = EA[u_x + \frac{1}{2}v_x^2 + \frac{1}{2}w_x^2] + \bar{F} \tag{6.11}$$

Where $\bar{F}$ contains all the additional higher order terms, which can be neglected. The simplified expression for $\bar{V}_x$ can be written as

$$\bar{V}_x \approx EA[u_x + \frac{1}{2}v_x^2 + \frac{1}{2}w_x^2] \tag{6.12}$$

Neglecting all the higher order terms and also the time derivative terms, the distributed inertia force $\bar{Z}_u$ (given in Appendix A) can be written as:

$$\bar{Z}_u = C_1[(n-1)l_e + x] + C_2 \tag{6.13}$$

where

$$\begin{aligned} C_1 &= 1 - (\beta_d \cos\theta_I + \beta_s \sin\theta_I) \\ C_2 &= (e_1 + e_2) - a(-\beta_d \sin\theta_I + \beta_s \cos\theta_I) \end{aligned}$$

Integrating Equation 6.10,

$$\frac{E_o}{m\Omega^2}\bar{V}_x = \int_{nl_e - x}^{1} \bar{Z}_u dx \tag{6.14}$$

$\bar{V}_x$ can be written as

$$\bar{V}_x = f(\xi) = a_1\xi^2 + a_2\xi + a_3 \tag{6.15}$$

where

$$\xi = \frac{x}{l_e}$$

a_1, a_2, a_3 are given as:

$$
\begin{aligned}
a_1 &= \frac{m\Omega^2}{E_o}[-\frac{1}{2}C_1 l_e^2] \\
a_2 &= \frac{m\Omega^2}{E_o}[-C_1(X_n - l_e) - C_2]l_e \\
a_3 &= \frac{m\Omega^2}{E_o}[\frac{1}{2}C_1\{X_N + (X_n - l_e)\} + C_2][X_N + (X_n - l_e)]
\end{aligned}
$$

where

$$X_N = \sum_{j=1}^{N}(l_e)_j = Nl_e$$

$$X_n = \sum_{j=1}^{n}(l_e)_j = nl_e$$

Combining Equation 6.15 with 6.12, the axial strain at the elastic axis can be written in terms of axial inertia force as

$$[u_x + \frac{1}{2}v_x^2 + \frac{1}{2}w_x^2] = \frac{f(\xi)}{EA} \tag{6.16}$$

Using Equation 6.16, the underlined terms in $\left\{F_1^E\right\}$,$\left\{F_2^E\right\}$ and $\left\{F_3^E\right\}$ are modified to a set of linear stiffness matrices, which are given as:

$$\int_0^{l_e} \bar{V}_x v_x \{\Phi_c'\} \, dx = l_e \int_0^1 f(\xi) \{\Phi_c'\} \{\Phi_c'\}^T d\xi \{V\} = \left[K_{11}^{E'}\right] \{V\}$$

$$\int_0^{l_e} \bar{V}_x w_x \{\Phi_c'\} \, dx = l_e \int_0^1 f(\xi) \{\Phi_c'\} \{\Phi_c'\}^T d\xi \{W\} = \left[K_{22}^{E'}\right] \{W\}$$

$$\int_0^{l_e} \bar{T}_x \phi_x \left\{\Phi_q'\right\} dx = \left[\frac{EAC_o}{EA}\right] l_e \int_0^1 f(\xi) \left\{\Phi_q'\right\} \left\{\Phi_q'\right\}^T d\xi \{\Phi\} = \left[K_{33}^{E'}\right] \{\Phi\}$$

Combining these submatrices, a linear stiffness matrix $\left[K^{E'}\right]$ is obtained, which can be written as

$$
\left[K^{E'}\right] = \begin{bmatrix}
\left[K_{11}^{E'}\right] & [O] & [O] & [O] \\
[O] & \left[K_{22}^{E'}\right] & [O] & [O] \\
[O] & [O] & \left[K_{33}^{E'}\right] & [O] \\
[O] & [O] & [O] & [O]
\end{bmatrix} \tag{6.17}
$$

where

$$\left[K_{11}^{E'}\right] = \left[K_{22}^{E'}\right] = (a_1 + a_2 + a_3)[A_1] - (2a_1 + a_2)[A_2] + (2a_1)[A_3]$$

$$\left[K_{33}^{E'}\right] = (a_1 + a_2 + a_3)[B_1] - (2a_1 + a_2)[B_2] + (2a_1)[B_3]$$

where

$$[A_1] = \frac{1}{30l_e}\begin{bmatrix} 36 & 3l_e & -36 & 3l_e \\ 3l_e & 4l_e^2 & -3l_e & -l_e^2 \\ -36 & -3l_e & 36 & -3l_e \\ 3l_e & -l_e^2 & -3l_e & 4l_e^2 \end{bmatrix} \tag{6.18}$$

$$[A_2] = \frac{1}{60l_e}\begin{bmatrix} 36 & 0 & -36 & 6l_e \\ 0 & 6l_e^2 & 0 & -l_e^2 \\ -36 & 0 & 36 & -6l_e \\ 6l_e & -l_e^2 & -6l_e & 2l_e^2 \end{bmatrix} \tag{6.19}$$

$$[A_3] = \frac{1}{140l_e}\begin{bmatrix} 24 & -2l_e & -24 & 5l_e \\ -2l_e & 6l_e^2 & 2l_e & -l_e^2 \\ -24 & 2l_e & 24 & -5l_e \\ 5l_e & -l_e^2 & -5l_e & \frac{4}{3}l_e^2 \end{bmatrix} \tag{6.20}$$

$$[B_1] = \frac{EAC_o}{3EAl_e}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix} \tag{6.21}$$

$$[B_2] = \frac{EAC_o}{6EAl_e} \begin{bmatrix} 11 & -12 & 1 \\ -12 & 16 & -4 \\ 1 & -4 & 3 \end{bmatrix} \tag{6.22}$$

$$[B_2] = \frac{EAC_o}{30EAl_e} \begin{bmatrix} 23 & -26 & 3 \\ -26 & 32 & -6 \\ 3 & -6 & 3 \end{bmatrix} \tag{6.23}$$

Chapter 7

MODEL VALIDATION

The first step in any aeroelastic response and stability analysis is the evaluation of natural frequencies of the rotor blade. Using the inertial and structural model developed in this study, a structural dynamic analysis was performed. It may be noted that the inertial and the structural operators given in Equation 6.6 and Equation 6.8 respectively, are nonlinear. Since the structural dynamic analysis requires only linear terms, all the nonlinear terms are neglected. The corresponding linear equation for one beam finite element can be written as:

$$[M]_i\{\ddot{q}\}_i + [K]_i\{q\}_i = 0 \tag{7.1}$$

where $[M]_i$ represents the mass matrix of i^{th} element, (given in Equation 6.6. and in Sec. 6.2.1). The stiffness matrix $[K]_i$ consists of three components. They are $[K^{CF}]$ (given in Sec. 6.2.3) , $[K^E]$ (given in Sec. 6.3.1) and $[K^{E'}]$ (given in Equation 6.17). The element stiffness matrix is given as:

$$[K]_i = [K^{CF}] + [K^E] + [K^{E'}] \tag{7.2}$$

The element matrices are assembled to form the global finite element model for the rotor blade. Imposing the root boundary conditions, the corresponding rows and columns from the global model are eliminated. The resulting matrix equation can

be written as:

$$[M]\{\ddot{q}\} + [K]\{q\} = 0 \tag{7.3}$$

Performing an eigenanalysis, the natural frequencies of an undamped rotating blade in vacuum can be evaluated.

7.1 RESULTS

A structural dynamic analysis was carried out for a uniform, untwisted blade, for which the data and the uncoupled natural frequencies are available in Ref. [27]. The blade data corresponding to two rotor blade configurations are given in Table 1. The terminology *Soft-in-plane blade configuration* indicates that the first nondimensional lag frequency is less than 1. In *Stiff-in-plane blade configuration*, the first nondimensional lag frequency is greater than 1.

Using the data given in Table 1, the natural frequencies of a rotating blade are calculated. The results corresponding to soft-in-plane and stiff-in-plane blades are shown in Tables 2 and 3 respectively. It can be seen from these results that in the first flap, lag and torsional modes there is very a good convergence with respect to number of elements. The converged results, for first flap, first lag and first torsion are in excellent agreement with the results obtained in Ref. [27].

Soft-in-plane blade data	Stiff-in-plane blade data
$Im_{\zeta\zeta}/ml^2 = 0.0004$	$Im_{\zeta\zeta}/ml^2 = 0.0004$
$Im_{\eta\eta}/ml^2 = 0.0$	$Im_{\eta\eta}/ml^2 = 0.0$
$\theta_G = 0.0$	$\theta_G = 0.0$
$m = 1$	$m = 1$
$\beta_s = 0.0$	$\beta_s = 0.0$
$\beta_d = 0.0$	$\beta_d = 0.0$
$\beta_p = 0.0$	$\beta_p = 0.0$
$\theta_I = 0.0$	$\theta_I = 0.0$
$GJ/m\Omega^2 l^4 = 0.001473$	$GJ/m\Omega^2 l^4 = 0.001473$
$EA/m\Omega^2 l^2 = 20.0$	$EA/m\Omega^2 l^2 = 20.0$
$e_1 = 0.0$	$e_1 = 0.0$
$e_2 = 0.0$	$e_2 = 0.0$
$a = 0.0$	$a = 0.0$
$EAC_o/EA = 0.00021036$	$EAC_o/EA = 0.0008166$
$EI_{\zeta\zeta}/m\Omega^2 l^4 = 0.0301$	$EI_{\zeta\zeta}/m\Omega^2 l^4 = 0.1474$
$EI_{\eta\eta}/m\Omega^2 l^4 = 0.0106$	$EI_{\eta\eta}/m\Omega^2 l^4 = 0.0106$

Offsets of mass centre and tension centre from elastic centre are zero.

TABLE 1. Input data for soft-in-plane and stiff-in-plane blade

Mode	Uncoupled Natural Frequencies					
	No.of Element					Ref.[27]
	N=1	N=2	N=3	N=4	N=5	
1st Lag	.7605	.7363	.7331	.7326	.7326	0.732
2nd Lag	6.4123	4.4947	4.4662	4.4587	4.4563	–
1st Flap	1.1507	1.1301	1.1261	1.1250	1.1247	1.125
2nd Flap	4.4102	3.4418	3.4165	3.4109	3.4089	–
3rd Flap	–	8.8800	7.7140	7.6605	7.6376	–
1st Torsion	3.2753	3.2640	3.2634	3.2632	3.2632	3.263
1st Axial	6.9798	6.9551	6.9538	6.9533	6.9533	–

TABLE 2. Results for Soft-in-plane blade

Mode	Uncoupled Natural Frequencies					
	No.of Element					Ref.[27]
	N=1	N=2	N=3	N=4	N=5	
1st Lag	1.4302	1.4174	1.4165	1.4171	1.4166	1.417
2nd Lag	13.414	8.7951	8.7478	8.7308	8.7258	–
1st Flap	1.1507	1.1301	1.1261	1.1250	1.1247	1.125
2nd Flap	4.4102	3.4418	3.4165	3.4109	3.4089	–
3rd Flap		8.8800	7.7140	7.6605	7.6376	–
1st Torsion	3.5167	3.5023	3.5015	3.5014	3.5013	3.501
1st Axial	6.9798	6.9551	6.9538	6.9533	6.9533	–

TABLE 3. Results for Stiff-in-plane blade

Chapter 8

CONCLUDING REMARKS

A general structural dynamic model of a helicoptor rotor blade has been formulated. This model consists of various geometrical complexities of the rotor system and also the rigid body motion of the hub. The rotor blade is modelled using beam finite elements having 14 degrees of freedom. The equations of the motion are derived using the Hamilton's principle. Various linear and nonlinear terms associated with inertial and structural operators have been derived in a matrix form which is suitable for application to further studies on rotor blade aeroelasticity and helicoptor vibration.

The analytical model developed in this study was used to calculate the uncoupled natural frequencies of an isotropic, untwisted and hingeless rotor blade. The results of the analysis are found to be in excellent agreement with the results available in the literature.

8.1 SCOPE FOR FURTHER WORK

Rotor blade aerodynamic model has to be formulated and integrated with the present structural dynamic model. The resulting analytical equations can be used for the study of aeroelastic response and stability analysis of a rotor blade, coupled rotor-fuselage stability analysis and vibration analysis of helicopters.

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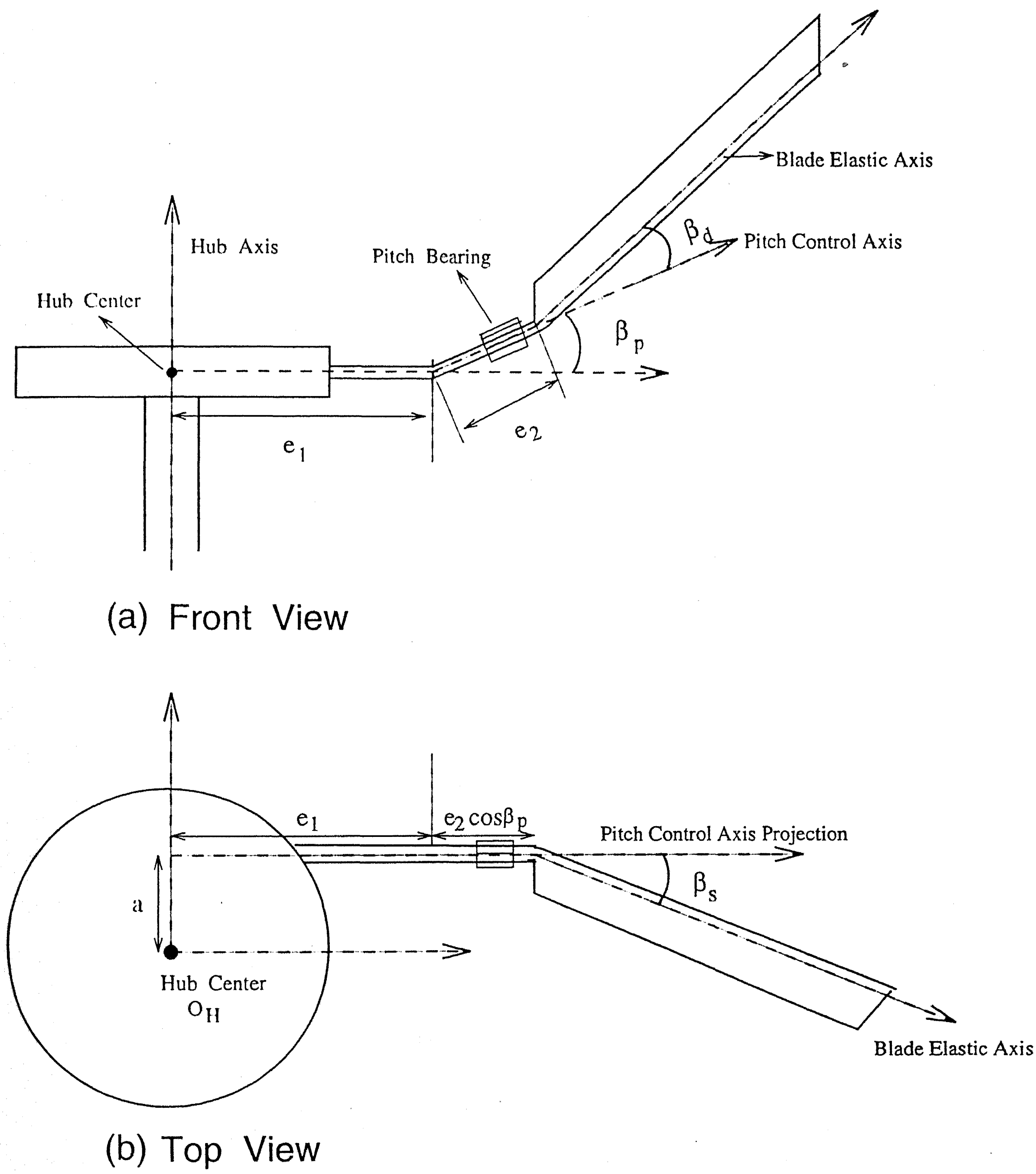

Fig. 1. Rotor Blade Configuration

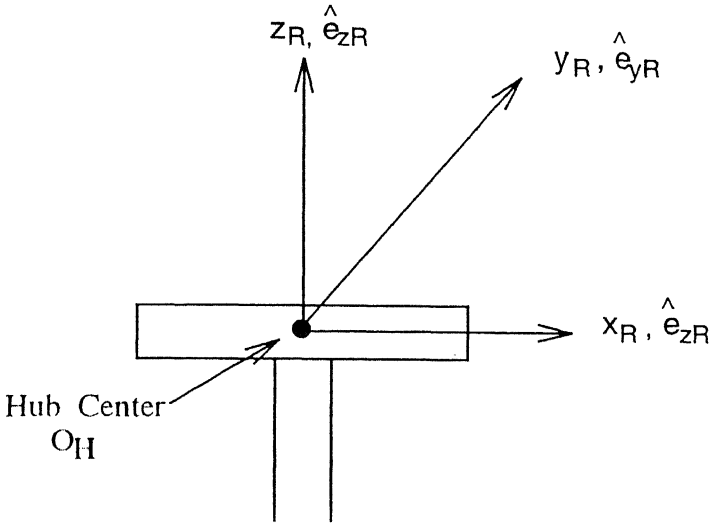

Fig. 2. Inertial System - R

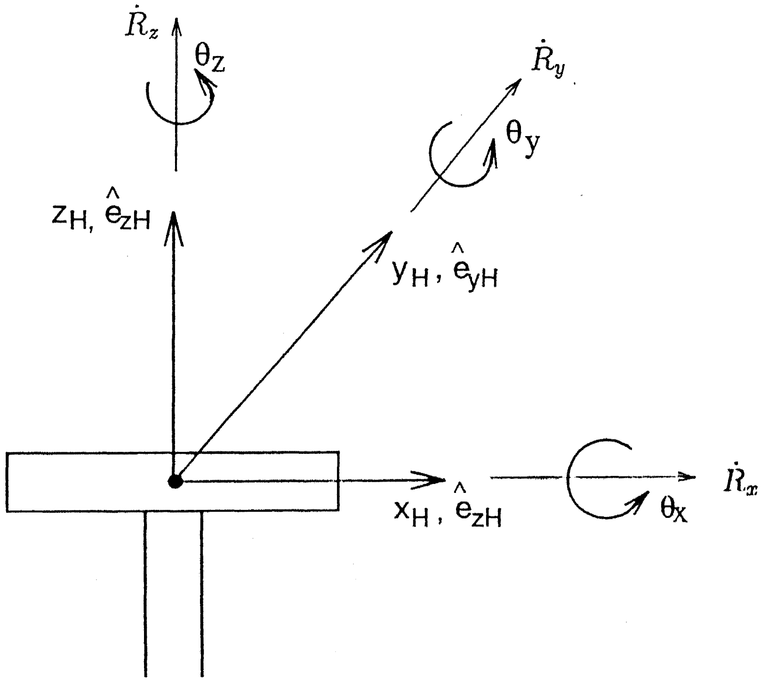

Fig.3. Body Fixed Hub System - H

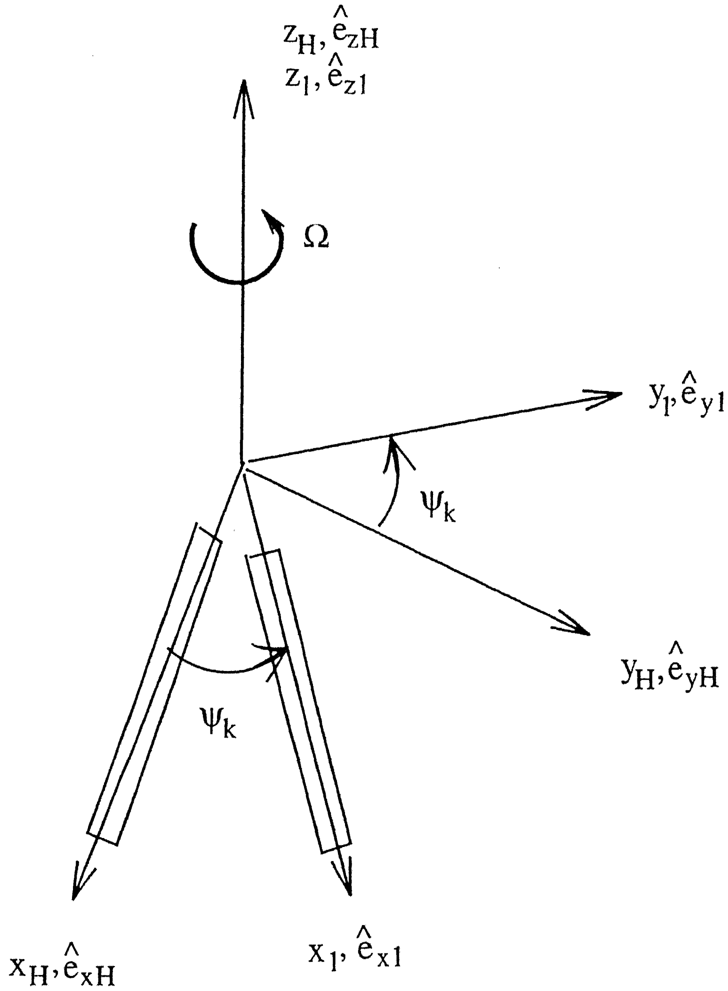

Fig. 4. Rotating Hub System - 1

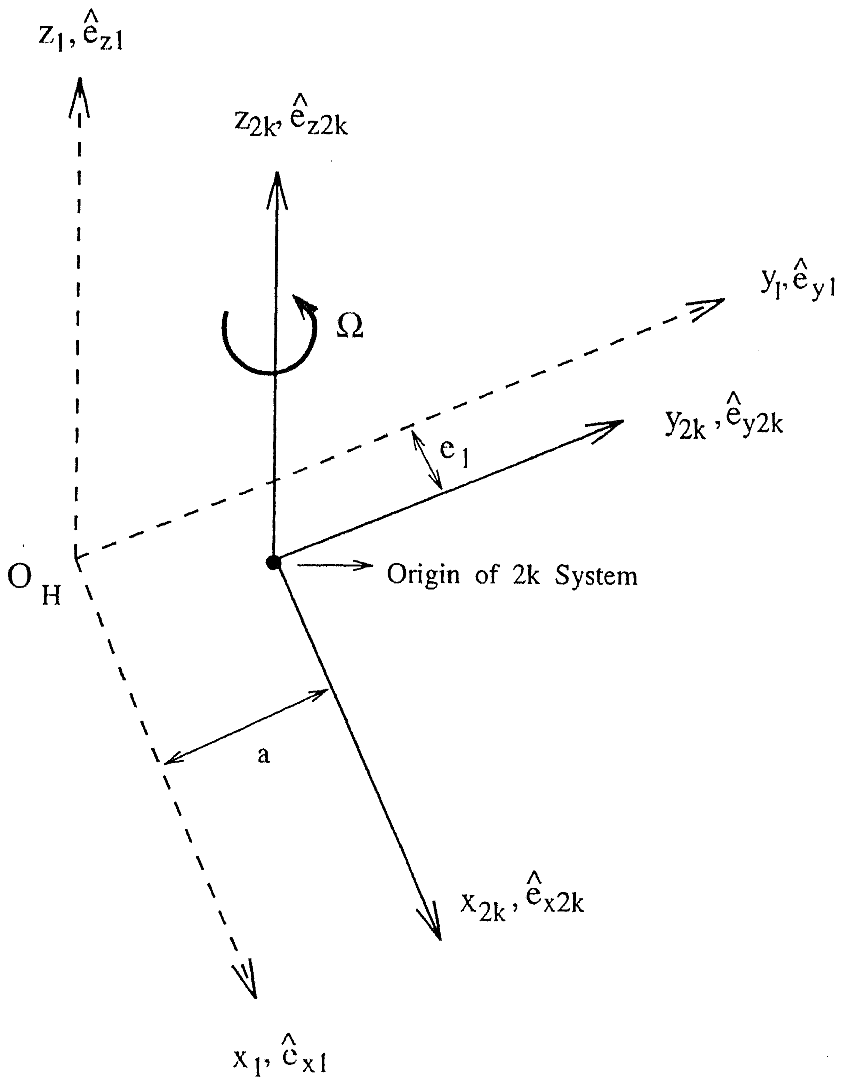

Fig. 5. Rotating Hub System - 2k

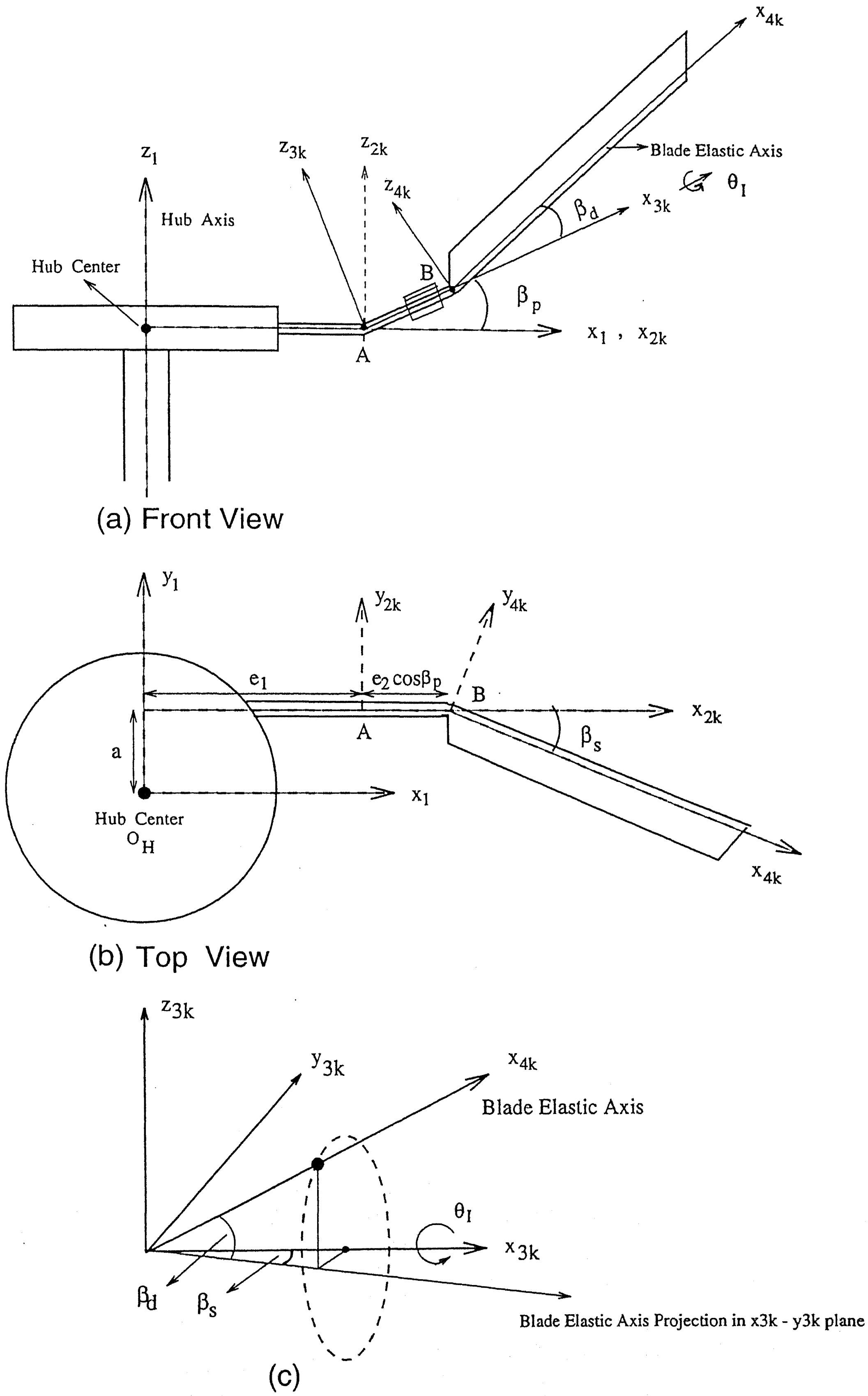

Fig. 6. Blade Coordinate Systems 3k and 4k

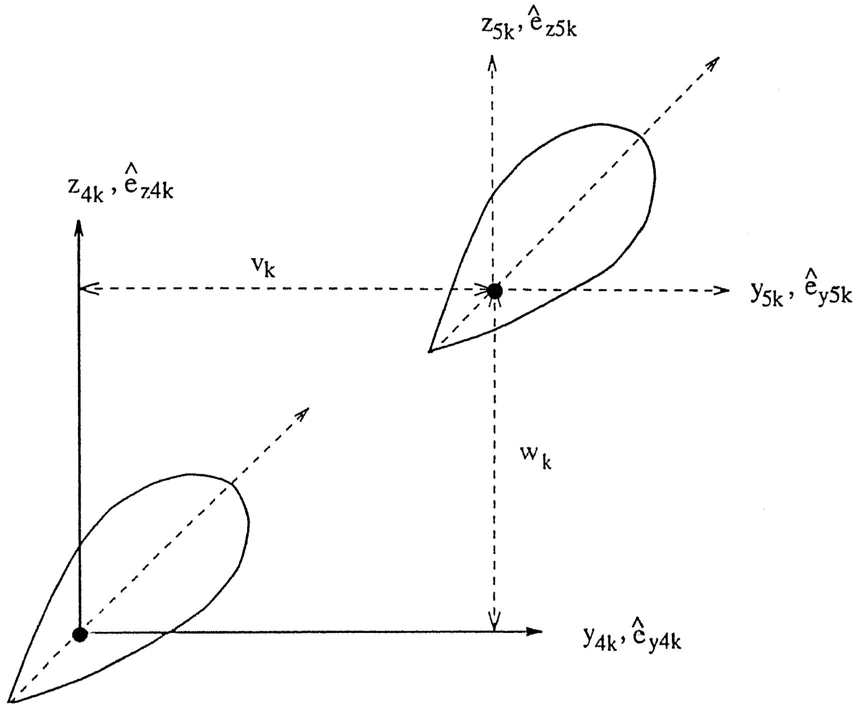

Fig.7. Rotating Blade Fixed System - 5k

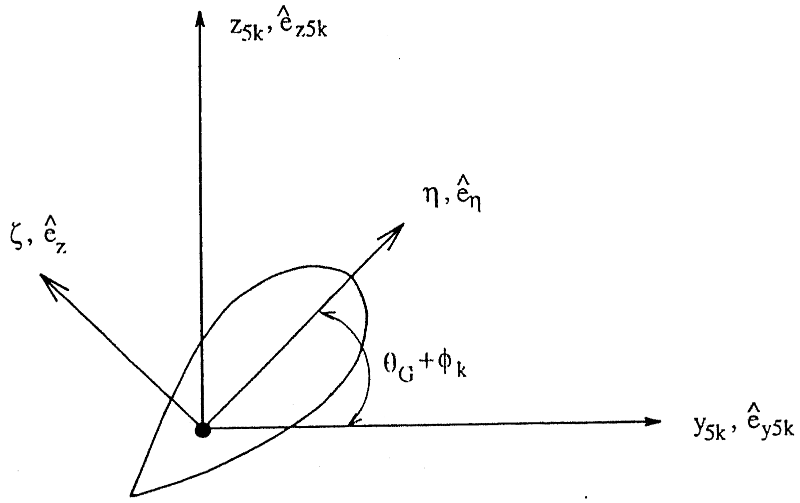

Fig.8. Cross-sectional Principal Coordinate System

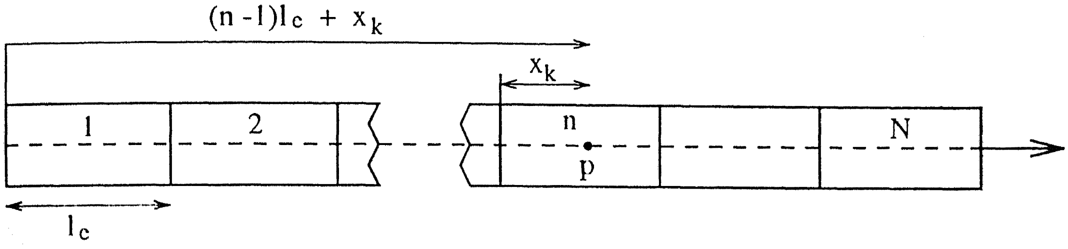

Fig.9 Finite Element Model of Blade

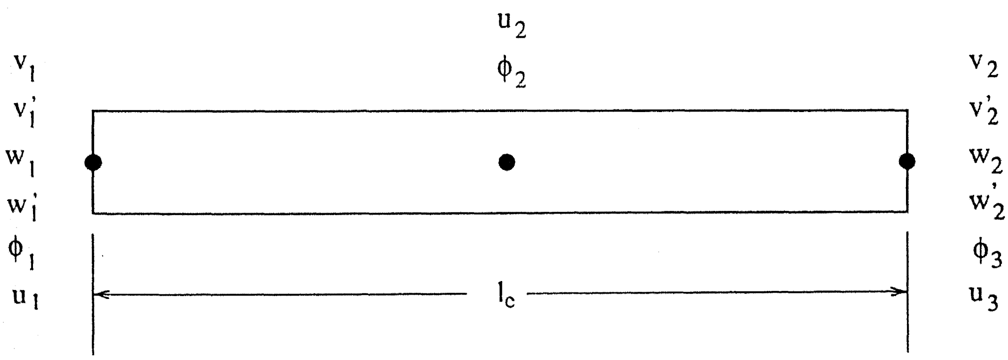

Fig.10 Element Nodal Degrees of Freedom

Appendix A

TERMS ASSOCIATED WITH KINETIC ENERGY VARIATION

$\bar{Z}_u =$

$$[m\eta_m \sin(\theta_G + \phi_k) + m\zeta_m \cos(\theta_G + \phi_k)][(-\dot{\phi}_k + w'_k)$$

$$\{\dot{\theta}_z - \dot{\theta}_I(\beta_d \cos\theta_I + \beta_s \sin\theta_I) - (\cos\theta_I + \dot{\theta}_I)\} - (\cos\theta_I + \dot{\theta}_I)\{\dot{\theta}_I w'_k(\beta_d \cos\theta_I$$

$$+\beta_s \sin\theta_I)\} + \dot{\theta}_I w'_k \cos\theta_I\{\dot{\theta}_x \sin\psi_k - \dot{\theta}_y \cos\psi_k$$

$$+\dot{\theta}_I(\beta_d \sin\theta_I - \beta_s \cos\theta_I)\}]$$

$$+[m\eta_m \sin(\theta_G + \phi_k) - m\zeta_m \cos(\theta_G + \phi_k)][\{\dot{\theta}_I(\beta_d < \cos\theta_I + \sin\theta_I >$$

$$+\beta_s < \sin\theta_I - \cos\theta_I > +1) + (\beta_p + \beta_d \cos\theta_I + \beta_s \sin\theta_I)\}$$

$$\{\dot{\theta}_z - \dot{\theta}_I(\beta_d \cos\theta_I + \beta_s \sin\theta_I) - (\cos\theta_I + \dot{\theta}_I)\} - \{\dot{\theta}_I(\beta_d \cos\theta_I + \beta_s \sin\theta_I)$$

$$+(\cos\theta_I + \dot{\theta}_I)\} \cos\theta_I(\dot{\theta}_x \cos\psi_k + \dot{\theta}_y \sin\psi_k) - (\cos\theta_I + \dot{\theta}_I)\cos\dot{\theta}_I \theta_z w'_k]$$

$$+[m\eta_m \cos(\theta_G + \phi_k) + m\zeta_m \sin(\theta_G + \phi_k)][\{-\dot{\phi}_k + \dot{\theta}_I(\beta_d < \cos\theta_I + \sin\theta_I >$$

$$+\beta_s < \sin\theta_I - \cos\theta_I > +1) + (\beta_p + \beta_d \cos\theta_I + \beta_s \sin\theta_I)\}\{\dot{\theta}_x \sin\psi_k - \dot{\theta}_y \cos\psi_k$$

$$+\dot{\theta}_I(\beta_d \sin\theta_I - \beta_s \cos\theta_I - 1) + (1+\dot{\theta}_z)\sin\theta_I\}$$
$$+\{\dot{\theta}_I(\beta_d \sin\theta_I - \beta_s\cos\theta_I - 1 + \sin\theta_I)(\dot{\theta}_x \cos\psi_k + \dot{\theta}_y \sin\psi_k)\}]$$
$$+[m\eta_m \cos(\theta_G+\phi_k) - m\zeta_m \sin(\theta_G+\phi_k)][v_k'(\dot{\theta}_I \cos\theta_I - 1)\{\dot{\theta}_z - \dot{\theta}_I(\beta_d\cos\theta_I$$
$$+\beta_s \sin\theta_I) - (\cos\theta_I + \dot{\theta}_I)\} + (\cos\theta_I + \dot{\theta}_I)\{\dot{\theta}_z \cos\theta_I v_k' - \dot{\theta}_I v_k'(\beta_d \cos\theta_I + \beta_s \sin\theta_I)$$
$$+\dot{\theta}_I v_k' w_k'\} + \dot{\theta}_I v_k'\{\dot{\theta}_x \sin\psi_k - \dot{\theta}_y \cos\psi_k + \dot{\theta}_I(\beta_d \sin\theta_I - \beta_s \cos\theta_I)\}]$$
$$-m[\dot{\theta}_I w_k + (e_1 + e_2 + < n-1 > l_e + x_k)(1 - \dot{\theta}_I)]$$
$$[\dot{\theta}_z - \dot{\theta}_I(\beta_d \cos\theta_I + \beta_s \sin\theta_I) - (\cos\theta_I + \dot{\theta}_I)]$$
$$+m[\cos\theta_I(\dot{R}_x \sin\psi_k - \dot{R}_y\cos\psi_k) - (\dot{\theta}_z + \beta_d\cos\theta_I + \beta_s\sin\theta_I)(< n-1 > l_e + x_k)]$$
$$[\dot{\theta}_I(\beta_d\cos\theta_I + \beta_s\sin\theta_I) + (\cos\theta_I + \dot{\theta}_I)]$$
$$-m[(\dot{R}_x\cos\psi_k + \dot{R}_y\sin\psi_k)(\beta_s\cos\theta_I - \beta_d\sin\theta_I) -$$
$$\dot{\theta}_I(\beta_s\sin\theta_I + \beta_d\cos\theta_I)(e_1+e_2) - w_k(\dot{\theta}_x\cos\psi_k + \dot{\theta}_y\sin\psi_k)$$
$$+\dot{\theta}_z\cos\theta_I(e_1+e_2) - \dot{\theta}_I\{2(\beta_d\cos\theta_I + \beta_s\sin\theta_I)(e_1+e_2)$$
$$+a(\beta_s\cos\theta_I - \beta_d\sin\theta_I) - u_k\} - w_k(\beta_p + \beta_d\cos\theta_I + \beta_s\sin\theta_I)$$
$$+u_k + a(\beta_s\cos\theta_I - \beta_d\sin\theta_I)\cos\theta_I][\cos\theta_I + \dot{\theta}_I]$$
$$+m[\{w_k + a\cos\theta_I + v_k - (\beta_s\cos\theta_I - \beta_d\sin\theta_I + 1)$$
$$(< n-1 > l_e + x_k) - (e_1+e_2)\}$$
$$\{\dot{\theta}_x\sin\psi_k - \dot{\theta}_y\cos\psi_k + \dot{\theta}_I(\beta_d\sin\theta_I - \beta_s\cos\theta_I)\} + \{\dot{R}_x$$
$$+(\dot{\theta}_x\sin\psi_k - \dot{\theta}_y\cos\psi_k)(< n-1 > l_e + x_k)\}$$
$$\{\dot{\theta}_I(\beta_d\cos\theta_I - \beta_s\sin\theta_I - 1)\sin\theta_I\}]$$
$$+m[\ddot{R}_x\{\cos\psi_k - \sin\psi_k(-\beta_s\cos\theta_I + \beta_d\sin\theta_I)\} + \dot{R}_x\{-\sin\psi_k$$
$$-\cos\psi_k(-\beta_s\cos\theta_I + \beta_d\sin\theta_I) - \sin\psi_k\dot{\theta}_I(\beta_s\sin\theta_I + \beta_d\cos\theta_I)\}]$$
$$-m[\ddot{R}_y\{\sin\psi_k + \cos\psi_k(-\beta_s\cos\theta_I + \beta_d\sin\theta_I)\} + \dot{R}_y\{\cos\psi_k$$
$$-\sin\psi_k(-\beta_s\cos\theta_I + \beta_d\sin\theta_I) + \cos\psi_k\dot{\theta}_I(\beta_s\sin\theta_I + \beta_d\cos\theta_I)\}]$$
$$-m[\ddot{R}_z(\beta_p + \beta_d\cos\theta_I + \beta_s\sin\theta_I) + \dot{R}_z\dot{\theta}_I(-\beta_d\sin\theta_I + \beta_s\cos\theta_I)]$$

$$-m[a\ddot{\theta}_I(\beta_s \sin\theta_I + \beta_d \cos\theta_I) + a\dot{\theta}_I^2(\beta_s \cos\theta_I - \beta_d \sin\theta_I) + \ddot{u}_k]$$
$$-m\eta_m[-\ddot{v}_k' \cos(\theta_G + \phi_k) + v_k'\ddot{\phi}_k \sin(\theta_G + \phi_k) + v_k'\dot{\phi}_k^2 \cos(\theta_G + \phi_k)$$
$$+2\dot{v}_k'\dot{\phi}_k \sin(\theta_G + \phi_k) - \ddot{w}_k' \sin(\theta_G + \phi_k) - 2\dot{w}_k'\dot{\phi}_k \cos(\theta_G + \phi_k)$$
$$-w_k'\ddot{\phi}_k \cos(\theta_G + \phi_k) + w_k'\dot{\phi}_k^2 \sin(\theta_G + \phi_k)\}]$$
$$-m\zeta_m[\ddot{v}_k' \sin(\theta_G + \phi_k) + v_k'\ddot{\phi}_k \cos(\theta_G + \phi_k) - v_k'\dot{\phi}_k^2 \sin(\theta_G + \phi_k)$$
$$+2\dot{v}_k'\dot{\phi}_k \cos(\theta_G + \phi_k) - \ddot{w}_k' \cos(\theta_G + \phi_k) + 2\dot{w}_k'\dot{\phi}_k \sin(\theta_G + \phi_k)$$
$$+w_k'\ddot{\phi}_k \sin\psi_k + w_k'\dot{\phi}_k^2 \cos(\theta_G + \phi_k)\}]$$
$$+\cos\theta_I[\{m\eta_m \sin(\theta_G + \phi_k) + m\zeta_m \cos(\theta_G + \phi_k)\}$$
$$\{(\ddot{\theta}_x + \dot{\theta}_y)\sin\psi_k + (\dot{\theta}_x - \ddot{\theta}_y)\cos\psi_k\}$$
$$-\dot{\phi}_k\{m\eta_m \cos(\theta_G + \phi_k) + m\zeta_m \sin(\theta_G + \phi_k)\}\{\dot{\theta}_x \sin\psi_k - \dot{\theta}_y \cos\psi_k\}]$$
$$+m\cos\theta_I[w_k\{(\ddot{\theta}_x + \dot{\theta}_y)\sin\psi_k + (\dot{\theta}_x - \ddot{\theta}_y)\cos\psi_k\}$$
$$+\dot{w}_k\{\dot{\theta}_x \sin\psi_k - \dot{\theta}_y \cos\psi_k\}]$$
$$[\ddot{\theta}_z\{m(a\cos\theta_I + v_k) + m\eta_m \cos(\theta_G + \phi_k) + m\zeta_m \sin(\theta_G + \phi_k)\}$$
$$+\dot{\theta}_z\{m\dot{v}_k - m\eta_m\dot{\phi}_k \sin(\theta_G + \phi_k) + m\zeta_m\dot{\phi}_k \cos(\theta_G + \phi_k)\}]$$
$$+\ddot{\theta}_I[\{m(e_1 + e_2 - w_k) + m\eta_m \sin(\theta_G + \phi_k) - m\zeta_m \cos(\theta_G + \phi_k)\}$$
$$\{\beta_s \cos\theta_I - \beta_d \sin\theta_I\} + \{m\eta_m \cos(\theta_G + \phi_k) + m\zeta_m \sin(\theta_G + \phi_k)\}$$
$$\{1 - (\beta_d \cos\theta_I + \beta_s \sin\theta_I)\} - m\eta_m \sin(\theta_G + \phi_k) + m\zeta_m \cos(\theta_G + \phi_k)$$
$$+m\{a\cos\theta_I + v_k + w_k + (e_1 + e_2 - a\cos\theta_I - v_k)(\beta_d \cos\theta_I + \beta_s \sin\theta_I)\}]$$
$$+\dot{\theta}_I \cos\theta_I[\{-m\dot{w}_k + m\eta_m\dot{\phi}_k \cos(\theta_G + \phi_k) + m\zeta_m\dot{\phi}_k \sin(\theta_G + \phi_k)\}\{\beta_s \cos\theta_I$$
$$-\beta_d \sin\theta_I\} - \dot{\theta}_I\{m(e_1 + e_2 - w_k) + m\eta_m \sin(\theta_G + \phi_k) - m\zeta_m \cos(\theta_G + \phi_k)\}$$
$$\{\beta_s \sin\theta_I - \beta_d \cos\theta_I\} + \{m\dot{v}_k - m\eta_m\dot{\phi}_k \sin(\theta_G + \phi_k) + m\zeta_m\dot{\phi}_k \cos(\theta_G + \phi_k)\}$$
$$\{(1 - \beta_d \cos\theta_I - \beta_s \sin\theta_I) + \dot{\theta}_I\{m(e_1 + e_2 - a - w_k) - m\eta_m \cos(\theta_G + \phi_k)$$
$$-m\zeta_m \sin(\theta_G + \phi_k)\} + m\dot{w}_k - m\eta_m\dot{\phi}_k \cos(\theta_G + \phi_k) - m\zeta_m\dot{\phi}_k \sin(\theta_G + \phi_k))]$$
$$+\cos\theta_I[m\{(e_1 + e_2)\dot{\theta}_I(\beta_s \sin\theta_I + \beta_d \cos\theta_I) - a\dot{\theta}_I \sin\theta_I) - \dot{v}_k\}$$

$$+m\eta_m\dot{\phi}_k \sin(\theta_G + \phi_k) - m\zeta_m\dot{\phi}_k \cos(\theta_G + \phi_k)]$$

$$+\sin\theta_I[m\dot{w}_k - m\eta_m\dot{\phi}_k \cos(\theta_G + \phi_k) - m\zeta_m\dot{\phi}_k \sin(\theta_G + \phi_k)]$$

$$+(-\dot{\theta}_I + \sin\theta_I)\sin\theta_I\{-w_k'\{m\eta_m \sin(\theta_G + \phi_k) - m\zeta_m \cos(\theta_G + \phi_k)\}$$

$$-v_k'\{m\eta_m \cos(\theta_G + \phi_k) - m\zeta_m \sin(\theta_G + \phi_k)\}$$

$$+m\{e_1 + e_2 + < n - 1 > l_e + x_k + u_k\} - ma\dot{\theta}_I \cos\theta_I]$$

$$\bar{Z}_v =$$

$$[\{m\eta_m \sin(\theta_G + \phi_k) + m\zeta_m \cos(\theta_G + \phi_k)\}\dot{w}'_k$$

$$+\{m\eta_m \cos(\theta_G + \phi_k) - m\zeta_m \sin(\theta_G + \phi_k)\}\dot{v}'_k$$

$$+\{m\eta_m \sin(\theta_G + \phi_k) - m\zeta_m \cos(\theta_G + \phi_k)\}\{\dot{\theta}_I(\beta_s \cos\theta_I - \beta_d \sin\theta_I - 1)\}$$

$$+\{m\eta_m \cos(\theta_G + \phi_k) + m\zeta_m \sin(\theta_G + \phi_k)\}$$

$$\{(\dot{\theta}_I(1 - \beta_d \cos\theta_I - \beta_s \sin\theta_I) + \cos\theta_I\}$$

$$+m\{(\dot{\theta}_I + 1)(v'_k + a < 2\cos\theta_I - \sin\theta_I >) + \dot{\theta}_I w_k\}][\dot{\theta}_z$$

$$+\dot{\theta}_I(1 + \beta_d \cos\theta_I + \beta_s \sin\theta_I) + \cos\theta_I]$$

$$+[\{m\eta_m \cos(\theta_G + \phi_k) + m\zeta_m \sin(\theta_G + \phi_k)\}(-\dot{\theta}_z)$$

$$+\{m\eta_m \sin(\theta_G + \phi_k) - m\zeta_m \cos(\theta_G + \phi_k)\}\{\dot{\theta}_x \sin\psi_k - \dot{\theta}_y \cos\psi_k) \cos\theta_I$$

$$+m(\dot{R}_x \cos\psi_k + \dot{R}_y \sin\psi_k)][\dot{\theta}_I(\beta_d \cos\theta_I + \beta_s \sin\theta_I) + (\cos\theta_I + \dot{\theta}_I)]$$

$$+[\{m\eta_m \sin(\theta_G + \phi_k) + m\zeta_m \cos(\theta_G + \phi_k)\}v'_k\dot{\phi}_k$$

$$-\{m\eta_m \cos(\theta_G + \phi_k) - m\zeta_m \sin(\theta_G + \phi_k)\}w'_k\dot{\phi}_k$$

$$+m\{(-\beta_s \cos\theta_I + \beta_d \sin\theta_I)(-\dot{R}_x \sin\psi_k + \dot{R}_y \cos\psi_k)$$

$$+\dot{R}_z(\beta_p + \beta_d \cos\theta_I + \beta_s \sin\theta_I) + a\dot{\theta}_I(\beta_s \sin\theta_I + \beta_d \cos\theta_I) + \dot{u}_k$$

$$-w_k(\dot{\theta}_x \sin\psi_k - \dot{\theta}_y \cos\psi_k) - \dot{\theta}_z(a + v_k)\}$$

$$-m\dot{\theta}_I\{(-w_k + e_1 + e_2)(\beta_s \cos\theta_I - \beta_d \sin\theta_I)$$

$$+(e_1 + e_2 - a\cos\theta_I - v_k)(\beta_d \cos\theta_I + \beta_s \sin\theta_I)\}$$

$$-m(e_1 + e_2)(\beta_s \cos\theta_I - \beta_d \sin\theta_I) \cos\theta_I$$

$$-\{mw_k - m\eta_m \sin(\theta_G + \phi_k) + m\zeta_m \cos(\theta_G + \phi_k)\} \sin\theta_I][\cos\theta_I + \dot{\theta}_I]$$

$$+[\{m\eta_m \cos(\theta_G + \phi_k) + m\zeta_m \sin(\theta_G + \phi_k)\}\{-\dot{\phi}_k$$

$$+\dot{\theta}_I(\beta_d < \cos\theta_I + \sin\theta_I > +\beta_s < \sin\theta_I - \cos\theta_I > +1)$$

$$+(\beta_p + \beta_d \cos\theta_I + \beta_s \sin\theta_I)\}$$

$$+\{m\eta_m\cos(\theta_G+\phi_k)-m\zeta_m\sin(\theta_G+\phi_k)\}\dot{\theta}_I v_k'$$

$$+\{m\eta_m\sin(\theta_G+\phi_k)+m\zeta_m\cos(\theta_G+\phi_k)\}\{\dot{\theta}_I w_k'$$

$$+m\{\dot{w}_k+a\cos\theta_I+v_k-(\beta_s\cos\theta_I-\beta_d\sin\theta_I+1)$$

$$(<n-1>l_e+x_k)-(e_1+e_2)\}]$$

$$[\dot{\theta}_x\cos\psi_k+\dot{\theta}_y\sin\psi_k+\dot{\theta}_I(\beta_d<\cos\theta_I+\sin\theta_I>+\beta_s<\sin\theta_I-\cos\theta_I>+1)]$$

$$+[\{m\eta_m\cos(\theta_G+\phi_k)+m\zeta_m\sin(\theta_G+\phi_k)\}\{\cos\theta_I(\dot{\theta}_x\cos\psi_k+\dot{\theta}_y\sin\psi_k)\}$$

$$+m\{\dot{R}_z\cos\theta_I+\dot{\theta}_x\sin\psi_k-\dot{\theta}_y\cos\psi_k)(<n-1>l_e+x_k)]$$

$$[\dot{\theta}_I(\beta_d<\cos\theta_I+\sin\theta_I>+\beta_s<\sin\theta_I-\cos\theta_I>+1)]$$

$$+[\{m\eta_m\cos(\theta_G+\phi_k)-m\zeta_m\sin(\theta_G+\phi_k)\}\{-v_k'(\dot{\theta}_x\sin\psi_k-\dot{\theta}_y\cos\psi_k)$$

$$+v_k'(\beta_s\cos\theta_I-\beta_d\sin\theta_I)\dot{\theta}_I\}$$

$$+\{m\eta_m\sin(\theta_G+\phi_k)+m\zeta_m\cos(\theta_G+\phi_k)\}\{-w_k'(\dot{\theta}_x\sin\psi_k-\dot{\theta}_y\cos\psi_k)$$

$$+w_k'(\beta_s\cos\theta_I-\beta_d\sin\theta_I)\dot{\theta}_I\}$$

$$-m\{(\dot{R}_x\cos\psi_k+\dot{R}_y\sin\psi_k)(\beta_p+\beta_d\cos\theta_I+\beta_s\sin\theta_I)+\dot{\theta}_I(\beta_d\sin\theta_I$$

$$-\beta_s\cos\theta_I)(e_1+e_2)+\dot{\theta}_x\cos\psi_k+\dot{\theta}_y\sin\psi_k)(a\cos\theta_I+v_k)+\dot{\theta}_x\sin\psi_k-\dot{\theta}_y\cos\psi_k)$$

$$(e_1+e_2)+a(\beta_p+\beta_d\cos\theta_I+\beta_s\sin\theta_I))\}+m\dot{\theta}_I\{(\beta_d<\cos\theta_I+\sin\theta_I>$$

$$+\beta_s<\sin\theta_I-\cos\theta_I>+1)(a+v_k)+u_k+a(\beta_s\cos\theta_I-\beta_d\sin\theta_I)\}$$

$$+m\dot{\theta}_I\sin\theta_I\{e_1+e_2+<n-1>l_e+x_k+u_k][\dot{\theta}_I]$$

$$-m\eta_m[\sin(\theta_G+\phi_k)\{-\ddot{\phi}_k+(-\dot{\theta}_x+\ddot{\theta}_y)\sin\psi_k+(\ddot{\theta}_x+\dot{\theta}_y)\cos\psi_k$$

$$-\ddot{\theta}_z w_k'\cos\theta_I-\dot{\theta}_z\dot{w}_k'\cos\theta_I+\ddot{\theta}_I(\beta_d<\cos\theta_I+\sin\theta_I>+\beta_s<\sin\theta_I-\cos\theta_I>)$$

$$+\dot{\theta}_I^2(\beta_d<-\sin\theta_I+\cos\theta_I>+\beta_s<\sin\theta_I+\cos\theta_I>)+\ddot{\theta}_I$$

$$+(\ddot{\theta}_I w_k'+\dot{\theta}_I\dot{w}_k')(\dot{\theta}_I<-\beta_d\sin\theta_I+\beta_s\cos\theta_I>+1)+\dot{\theta}_I(-\beta_d\sin\theta_I+\beta_s\cos\theta_I)$$

$$+\dot{w}_k'\cos\theta_I-\dot{\phi}_k(-\dot{\theta}_z v_k'\cos\theta_I+v_k'\dot{\theta}_I<\beta_d\cos\theta_I+\beta_s\sin\theta_I>+\dot{\theta}_I v_k'$$

$$+\dot{\theta}_I w_k' v_k'+v_k')\cos\theta_I\}+\cos(\theta_G+\phi_k)\{\dot{\phi}_k(-\dot{\phi}_k+\dot{\theta}_x\cos\psi_k+\dot{\theta}_y\sin\psi_k$$

$$-\dot{\theta}_z w_k'\cos\theta_I+\dot{\theta}_I\beta_d<\cos\theta_I+\sin\theta_I>+\dot{\theta}_I\beta_s<\sin\theta_I-\cos\theta_I>+\dot{\theta}_I$$

$$+\dot{\theta}_I w'_k < \beta_d \cos\theta_I + \beta_s \sin\theta_I > +\dot{\theta}_I w'_k + \beta_p + \beta_d \cos\theta_I$$

$$+\beta_s \sin\theta_I + w'_k \cos\theta_I) - \ddot{\theta}_z v'_k \cos\theta_I$$

$$-\dot{\theta}_z \dot{v}'_k \cos\theta_I + (\ddot{\theta}_I v'_k + \dot{\theta}_I \dot{v}'_k)(\dot{\theta}_I < -\beta_d \sin\theta_I + \beta_s \cos\theta_I > +1)$$

$$+\ddot{\theta}_I w'_k v'_k + \dot{\theta}_I (\dot{w}'_k v'_k + w'_k \dot{v}'_k) + \dot{v}'_k \cos\theta_I\}]$$

$$-m\zeta_m[\sin(\theta_G + \phi_k)\{\ddot{\theta}_z v'_k \cos\theta_I + \dot{\theta}_z \dot{v}'_k - (\ddot{\theta}_I v'_k + \dot{\theta}_I \dot{v}'_k)(\beta_d \cos\theta_I + \beta_s \sin\theta_I + 1)$$

$$-\ddot{\theta}_I v'_k w'_k - \dot{\theta}_I \dot{v}'_k w'_k - \dot{\theta}_I \dot{v}'_k \dot{w}'_k - \dot{v}'_k \cos\theta_I + \dot{\phi}_k(\dot{\phi}_k + \dot{\theta}_x \cos\psi_k + \dot{\theta}_y \sin\psi_k$$

$$-\dot{\theta}_z w'_k \cos\theta_I - \dot{\theta}_I \beta_d < \cos\theta_I + \sin\theta_I > -\dot{\theta}_I \beta_s < \sin\theta_I - \cos\theta_I > -\dot{\theta}_I$$

$$+\dot{\theta}_I w'_k < \beta_d \cos\theta_I + \beta_s \sin\theta_I > +\dot{\theta}_I w'_k$$

$$- < \beta_p + \beta_d \cos\theta_I + \beta_s \sin\theta_I > +w'_k \cos\theta_I)\}$$

$$+\cos(\theta_G + \phi_k)\{\dot{\phi}_k(\dot{\theta}_z v'_k \cos\theta_I - \dot{\theta}_I v'_k < \beta_d \cos\theta_I + \beta_s \sin\theta_I >$$

$$-v'_k \dot{\theta}_I - \dot{\theta}_I v'_k w'_k - v'_k \cos\theta_I)$$

$$-\ddot{\phi}_k + (\dot{\theta}_x - \ddot{\theta}_y) \sin\psi_k - (\ddot{\theta}_x + \dot{\theta}_y) \cos\psi_k + \ddot{\theta}_z w'_k \cos\theta_I + \dot{\theta}_z \dot{w}'_k \cos\theta_I$$

$$-\dot{\theta}_I^2(\beta_d < -\sin\theta_I + \cos\theta_I > +\beta_s < \cos\theta_I + \sin\theta_I >) - \ddot{\theta}_I(\beta_d < \cos\theta_I + \sin\theta_I >$$

$$+\beta_s < \sin\theta_I - \cos\theta_I > -\ddot{\theta}_I + \ddot{\theta}_I w'_k(\beta_d \cos\theta_I + \beta_s \sin\theta_I)$$

$$+\dot{\theta}_I \dot{w}'_k(\beta_d \cos\theta_I + \beta_s \sin\theta_I) + \dot{\theta}_I^2 w'_k(\beta_d \cos\theta_I + \beta_s \sin\theta_I)$$

$$+\ddot{\theta}_I w'_k + \dot{\theta}_I \dot{w}'_k - \dot{\theta}_I(-\beta_d \sin\theta_I + \beta_s \cos\theta_I) + \dot{w}'_k \cos\theta_I\}]$$

$$-m[(\ddot{R}_x + \dot{R}_y)\{-\sin\psi_k \cos\theta_I + \cos\psi_k(\beta_s \cos\theta_I - \beta_d \sin\theta_I)\}$$

$$+(-\dot{R}_x + \ddot{R}_y)\{\cos\psi_k \cos\theta_I + \sin\psi_k(\beta_s \cos\theta_I - \beta_d \sin\theta_I)\}$$

$$-\dot{\theta}_I(\dot{R}_x \cos\psi_k + \dot{R}_y \sin\psi_k)(\beta_s \cos\theta_I + \beta_d \cos\theta_I)$$

$$-\ddot{R}_z \sin\theta_I) - \dot{R}_z \dot{\theta}_I \cos\theta_I - a\ddot{\theta}_I \sin\theta_I) - a\dot{\theta}_I^2 \cos\theta_I$$

$$-\ddot{\theta}_I(\beta_s \cos\theta_I + \beta_d \cos\theta_I)(e_1 + e_2) + \ddot{v}_k - a\dot{\theta}_I \cos\theta_I$$

$$-\dot{\theta}_I^2(\beta_s \cos\theta_I - \beta_d \sin\theta_I)(e_1 + e_2) - \dot{w}_k(\dot{\theta}_x \cos\psi_k + \dot{\theta}_y \sin\psi_k)$$

$$-w_k\{(-\dot{\theta}_x + \ddot{\theta}_y) \sin\psi_k + (\ddot{\theta}_x + \dot{\theta}_y) \cos\psi_k\} - (\ddot{\theta}_I w_k + \dot{\theta}_I \dot{w}_k)\{\beta_d(\cos\theta_I + \sin\theta_I)$$

$$+\beta_s(\sin\theta_I - \cos\theta_I) + 1\} - \dot{\theta}_I^2 w_k\{\beta_d(-\sin\theta_I + \cos\theta_I) + \beta_s(\cos\theta_I + \sin\theta_I)\}$$

$$-\ddot{\theta}_I\{(\beta_d \cos\theta_I + \beta_s \sin\theta_I)(2 < e_1 + e_2 > + < n-1 > l_e + x_k + u_k)$$

$$+a(\beta_s \cos\theta_I - \beta_d \sin\theta_I)$$

$$-(e_1 + e_2) - (< n - 1 > l_e) - x_k - u_k\}$$

$$-\dot{\theta}_I\{\dot{\theta}_I(-\beta_d \sin\theta_I + \beta_s \cos\theta_I)(2 < e_1 + e_2 >$$

$$+ < n - 1 > l_e + x_k + u_k\} + (\beta_d \cos\theta_I + \beta_s \sin\theta_I)\dot{u}_k$$

$$-a\dot{\theta}_I(\beta_s \sin\theta_I + \beta_d \cos\theta_I) - \dot{u}_k\}$$

$$-\{\dot{w}_k(\beta_p + \beta_d \cos\theta_I + \beta_s \sin\theta_I) + w_k\dot{\theta}_I(-\beta_d \sin\theta_I + \beta_s \cos\theta_I)$$

$$+a \cos\theta_I)\dot{\theta}_I(\beta_s \sin\theta_I + \beta_d \cos\theta_I) + \dot{u}_k \cos\theta_I\}]$$

$$\bar{Z}_w =$$

$$[\{m\eta_m \sin(\theta_G + \phi_k) + m\zeta_m \cos(\theta_G + \phi_k)\}\dot{w}'_k$$

$$+\{m\eta_m \cos(\theta_G + \phi_k) - m\zeta_m \sin(\theta_G + \phi_k)\}\dot{v}'_k$$

$$+\{m\eta_m \sin(\theta_G + \phi_k) - m\zeta_m \cos(\theta_G + \phi_k)\}\{\dot{\theta}_I(\beta_s \cos\theta_I - \beta_d \sin\theta_I - 1)\}$$

$$+\{m\eta_m \cos(\theta_G + \phi_k) + m\zeta_m \sin(\theta_G + \phi_k)\}\{(\dot{\theta}_I(1 - \beta_d \cos\theta_I - \beta_s \sin\theta_I) + \cos\theta_I\}$$

$$+m\{(\dot{\theta}_I + 1)(v'_k + a < 2\cos\theta_I - \sin\theta_I >) + \dot{\theta}_I w_k\}][\dot{\theta}_x \sin\psi_k \cos\theta_I$$

$$-\dot{\theta}_y \cos\psi_k \cos\theta_I + \dot{\theta}_z \sin\theta_I - \dot{\theta}_I(\beta_s \cos\theta_I - \beta_d \sin\theta_I) + \dot{\theta}_I + \sin\theta_I]$$

$$+[\{m\eta_m \cos(\theta_G + \phi_k) + m\zeta_m \sin(\theta_G + \phi_k)\}(-\dot{\theta}_z)$$

$$+\{m\eta_m \sin(\theta_G + \phi_k) - m\zeta_m \cos(\theta_G + \phi_k)\}\{\dot{\theta}_x \sin\psi_k - \dot{\theta}_y \cos\psi_k) \cos\theta_I$$

$$+m(\dot{R}_x \cos\psi_k + \dot{R}_y \sin\psi_k)][\dot{\theta}_I(\beta_s \cos\theta_I - \beta_d \sin\theta_I) - \dot{\theta}_I - \sin\theta_I]$$

$$+[\{m\eta_m \sin(\theta_G + \phi_k) + m\zeta_m \cos(\theta_G + \phi_k)\}v'_k\dot{\phi}_k$$

$$-\{m\eta_m \cos(\theta_G + \phi_k) - m\zeta_m \sin(\theta_G + \phi_k)\}w'_k\dot{\phi}_k$$

$$+m\{(-\beta_s \cos\theta_I + \beta_d \sin\theta_I)(-\dot{R}_x \sin\psi_k + \dot{R}_y \cos\psi_k)$$

$$+\dot{R}_z(\beta_p + \beta_d \cos\theta_I + \beta_s \sin\theta_I) + a\dot{\theta}_I(\beta_s \sin\theta_I + \beta_d \cos\theta_I) + \dot{u}_k$$

$$-w_k(\dot{\theta}_x \sin\psi_k - \dot{\theta}_y \cos\psi_k) - \dot{\theta}_z(a + v_k)\}$$

$$-m\dot{\theta}_I\{(-w_k + e_1 + e_2)(\beta_s \cos\theta_I - \beta_d \sin\theta_I)$$

$$+(e_1 + e_2 - a\cos\theta_I - v_k)(\beta_d \cos\theta_I + \beta_s \sin\theta_I)\}$$

$$-m(e_1 + e_2)(\beta_s \cos\theta_I - \beta_d \sin\theta_I)\cos\theta_I$$

$$-\{mw_k - m\eta_m \sin(\theta_G + \phi_k) + m\zeta_m \cos(\theta_G + \phi_k)\}\sin\theta_I][-\dot{\theta}_I - \sin\theta_I]$$

$$+[\{m\eta_m \sin(\theta_G + \phi_k) + m\zeta_m \cos(\theta_G + \phi_k)\}(-\dot{\phi}_k + w'_k)$$

$$+\{m\eta_m \sin(\theta_G + \phi_k) - m\zeta_m \cos(\theta_G + \phi_k)\}\{\dot{\theta}_I(\beta_d < \cos\theta_I + \sin\theta_I >$$

$$+\beta_s < \sin\theta_I - \cos\theta_I > +1) + (\beta_p + \beta_d \cos\theta_I + \beta_s \sin\theta_I)\}$$

$$+\{m\eta_m \cos(\theta_G + \phi_k) - m\zeta_m \sin(\theta_G + \phi_k)\}\{v'_k(\dot{\theta}_I \cos\theta_I - 1)\}$$

$$-m\{\dot\theta_I w_k + (e_1 + e_2 + < n-1 > l_e + x_k)\}(1 - \dot\theta_I \cos\theta_I)][-\dot\theta_x \cos\psi_k - \dot\theta_y \sin\psi_k$$
$$-\dot\theta_I(\beta_d < \cos\theta_I + \sin\theta_I > +\beta_s < \sin\theta_I - \cos\theta_I > +1)$$
$$-(\beta_p + \beta_d \cos\theta_I + \beta_s \sin\theta_I)]$$
$$+[\{m\eta_m \sin(\theta_G + \phi_k) + m\zeta_m \cos(\theta_G + \phi_k)\}(\dot\theta_x \cos\psi_k + \dot\theta_y \sin\psi_k)$$
$$-m\{\dot R_x \cos\theta_I \sin\psi_k + \dot R_y \cos\theta_I \cos\psi_k$$
$$+(\dot\theta_z \cos\theta_I + \beta_d \cos\theta_I + \beta_s \sin\theta_I)(< n-1 > l_e + x_k)\}]$$
$$[\dot\theta_I(\beta_d < \cos\theta_I + \sin\theta_I > +\beta_s < \sin\theta_I - \cos\theta_I > +1) - (\beta_p + \beta_d \cos\theta_I + \beta_s \sin\theta_I)]$$
$$-[\{m\eta_m \cos(\theta_G + \phi_k) - m\zeta_m \sin(\theta_G + \phi_k)\}\{-\dot\theta_z v_k' \cos\theta_I + \dot\theta_I(\beta_d \cos\theta_I + \beta_s \sin\theta_I)v_k'$$
$$-\dot\theta_I v_k' w_k'\} + \{m\eta_m \sin(\theta_G + \phi_k) - m\zeta_m \cos(\theta_G + \phi_k)\}(-\dot\theta_z w_k')$$
$$+\{m\eta_m \sin(\theta_G + \phi_k) + m\zeta_m \cos(\theta_G + \phi_k)\}\{\dot\theta_I w_k'(\beta_d \cos\theta_I + \beta_s \sin\theta_I)\}$$
$$+m\{(\dot R_x \cos\psi_k + \dot R_y \sin\psi_k)(\beta_s \cos\theta_I - \beta_d \sin\theta_I) - \dot\theta_I(\beta_s \sin\theta_I + \beta_d \cos\theta_I)(e_1 + e_2)$$
$$-w_k(\dot\theta_x \cos\psi_k + \dot\theta_y \sin\psi_k) + \dot\theta_z \cos\theta_I(e_1 + e_2) - (\beta_p + \beta_d \cos\theta_I + \beta_s \sin\theta_I)w_k$$
$$+a\cos\theta_I(\beta_s \cos\theta_I - \beta_d \sin\theta_I)\} + m\dot\theta_I\{a\cos\theta_I(\beta_s \cos\theta_I - \beta_d \sin\theta_I) - u_k\}][\dot\theta_I]$$
$$-m[(-\ddot R_x \cos\psi_k + \dot R_x \sin\psi_k - \ddot R_y \sin\psi_k - \dot R_y \cos\psi_k)(\beta_p + \beta_d \cos\theta_I + \beta_s \sin\theta_I)$$
$$-\dot\theta_I(\dot R_x \cos\psi_k + \dot R_y \sin\psi_k)(-\beta_d \sin\theta_I + \beta_s \cos\theta_I) + \ddot R_z \cos\theta_I$$
$$+\ddot\theta_I(e_1 + e_2)(\beta_d \sin\theta_I - \beta_s \cos\theta_I) + \dot\theta_I^2(e_1 + e_2)(\beta_d \cos\theta_I + \beta_s \sin\theta_I) + \ddot w_k]$$
$$+m[\ddot R_x \sin\psi_k \sin\theta_I) - \dot R_x \cos\psi_k \sin\theta_I) + \dot R_x \dot\theta_I \sin\psi_k \cos\theta_I + \ddot R_y \cos\psi_k \sin\theta_I)$$
$$-\dot R_y \sin\psi_k \sin\theta_I - \dot R_y \dot\theta_I \cos\psi_k \cos\theta_I - a\ddot\theta_I \cos\theta_I + a\dot\theta_I^2 \sin\theta_I]$$
$$-[\{m\eta_m \cos(\theta_G + \phi_k) + m\zeta_m \sin(\theta_G + \phi_k)\}\{\ddot\theta_x \cos\psi_k - \dot\theta_x \sin\psi_k + \ddot\theta_y \sin\psi_k$$
$$+\dot\theta_y \cos\psi_k + \ddot\theta_I(\beta_d < \cos\theta_I + \sin\theta_I > +\beta_s < \sin\theta_I - \cos\theta_I > +1)$$
$$+\dot\theta_I^2(\beta_d < -\sin\theta_I + \cos\theta_I > +\beta_s < \cos\theta_I + \sin\theta_I >) + \dot\theta_I(-\beta_d \sin\theta_I + \beta_s \cos\theta_I)\}]$$
$$-[\{-m\eta_m \sin(\theta_G + \phi_k) + m\zeta_m \cos(\theta_G + \phi_k)\}][\dot\phi_k\{\dot\theta_x \cos\psi_k + \dot\theta_y \sin\psi_k$$
$$+\dot\theta_I(\beta_d < \cos\theta_I + \sin\theta_I > +\beta_s < \sin\theta_I - \cos\theta_I > +1) + \beta_p + \beta_d \cos\theta_I + \beta_s \sin\theta_I)\}]$$
$$-[\{m\eta_m \cos(\theta_G + \phi_k) - m\zeta_m \sin(\theta_G + \phi_k)\}\{\ddot\phi_k - (\ddot\theta_x \sin\psi_k + \dot\theta_x \cos\psi_k$$

$$-\ddot{\theta}_y \cos\psi_k + \dot{\theta}_y \sin\psi_k)v'_k - (\dot{\theta}_x \sin\psi_k - \dot{\theta}_y \cos\psi_k)\dot{v}'_k - \dot{\theta}_I v'_k(\beta_s \sin\theta_I + \beta_d \cos\theta_I)$$

$$+\dot{v}'_k(\beta_s \cos\theta_I - \beta_d \sin\theta_I + 1) - w'_k(\dot{\theta}_x \sin\psi_k - \dot{\theta}_y \cos\psi_k$$

$$- < \beta_s \cos\theta_I - \beta_d \sin\theta_I + 1 >)\}]$$

$$-[\{m\eta_m \sin(\theta_G + \phi_k) + m\zeta_m \cos(\theta_G + \phi_k)\}\{-\dot{\phi}_k + (\dot{\theta}_x \sin\psi_k - \dot{\theta}_y \cos\psi_k$$

$$- < \beta_s \cos\theta_I - \beta_d \sin\theta_I + 1 >)\dot{v}'_k - (\ddot{\theta}_x \sin\psi_k + \dot{\theta}_x \cos\psi_k - \ddot{\theta}_y \cos\psi_k + \dot{\theta}_y \sin\psi_k)w'_k$$

$$-(\dot{\theta}_x \sin\psi_k - \dot{\theta}_y \cos\psi_k)\dot{w}'_k - \dot{\theta}_I(\beta_s \sin\theta_I + \beta_d \cos\theta_I)w'_k$$

$$+(\beta_s \cos\theta_I - \beta_d \sin\theta_I + 1)\dot{w}'_k\}]$$

$$-m[\ddot{\theta}_x\{(a\cos\theta_I + v_k)\cos\psi_k + (e_1 + e_2)\sin\psi_k \cos\theta_I\} + \dot{\theta}_x\{-(a\cos\theta_I + v_k)\sin\psi_k$$

$$+(\dot{v}_k + e_1 + e_2)\cos\psi_k \cos\theta_I\} + \ddot{\theta}_y\{(a + v_k)\sin\psi_k - (e_1 + e_2)\cos\psi_k \cos\theta_I\}$$

$$+\dot{\theta}_y\{(\dot{v}_k + e_1 + e_2)\sin\psi_k + (a + v_k)\cos\psi_k \cos\theta_I\} + \ddot{\theta}_I\{(\beta_d < -\sin\theta_I + \cos\theta_I >$$

$$+\beta_s < \cos\theta_I + \sin\theta_I > +1)(a\cos\theta_I + v_k) - (e_1 + e_2) - (\beta_s \cos\theta_I - \beta_d \sin\theta_I + 1)$$

$$(< n - 1 > l_e + x_k + u_k)\} + \dot{\theta}_I\{\dot{\theta}_I(\beta_d < -\sin\theta_I + \cos\theta_I > +\beta_s < \cos\theta_I + \sin\theta_I >)$$

$$(a + v_k) + \dot{v}_k(\beta_d < \cos\theta_I + \sin\theta_I > +\beta_s < \sin\theta_I - \cos\theta_I > +1)$$

$$+\dot{\theta}_I(\beta_s \sin\theta_I + \beta_d \cos\theta_I)(< n - 1 > l_e + x_k + u_k) - \dot{u}_k(\beta_s \cos\theta_I - \beta_d \sin\theta_I + 1)\}$$

$$-a\dot{\theta}_I(\beta_s \sin\theta_I + \beta_d \cos\theta_I) + a\dot{\theta}_I(-\beta_d \sin\theta_I + \beta_s \cos\theta_I)]$$

$$-[\{m\eta_m \cos(\theta_G + \phi_k) - m\zeta_m \sin(\theta_G + \phi_k)\}\{v'_k\dot{\theta}_I \cos\theta_I + \dot{v}'_k \sin\theta_I\}]$$

$$-[\{m\eta_m \sin(\theta_G + \phi_k) + m\zeta_m \cos(\theta_G + \phi_k)\}\{w'_k\dot{\theta}_I \cos\theta_I + \dot{w}'_k \sin\theta_I\}]$$

$$+m\dot{\theta}_I \cos\theta_I[e_1 + e_2 + < n - 1 > l_e + x_k + u_k - a\dot{\theta}_I(\beta_s \cos\theta_I - \beta_d \sin\theta_I)$$

$$+a\ddot{\theta}_I \sin\theta_I)(\beta_s \cos\theta_I - \beta_d \sin\theta_I)]$$

$\bar{Z}_\phi =$

$$[-\dot{\phi}_k(Im_{\zeta\zeta} + Im_{\eta\zeta})\{w'_k \sin(\theta_G + \phi_k) + v'_k \cos(\theta_G + \phi_k)\}]$$

$$[\dot{\theta}_I\{\cos(\theta_G + \phi_k) - \sin(\theta_G + \phi_k)\} + \cos(\theta_G + \phi_k)]$$

$$[-\dot{\phi}_k(Im_{\eta\eta} + Im_{\eta\zeta})\{w'_k \sin(\theta_G + \phi_k) + v'_k \cos(\theta_G + \phi_k)\}]$$

$$[\dot{\theta}_I\{\sin(\theta_G + \phi_k) + \cos(\theta_G + \phi_k)\} + \sin(\theta_G + \phi_k)]$$

$$-Im_{\zeta\zeta}[\{\sin(\theta_G + \phi_k)(\dot{v}'_k + \dot{\theta}_z + \dot{\theta}_I < 1 - \beta_d \cos\theta_I - \beta_s \sin\theta_I > + \cos\theta_I)$$

$$+\cos(\theta_G + \phi_k)(-\dot{w}'_k + \dot{\theta}_x \sin\psi_k \cos\theta_I - \dot{\theta}_y \cos\psi_k \cos\theta_I$$

$$-\dot{\theta}_I < \beta_s \cos\theta_I - \beta_d \sin\theta_I + 1 >)$$

$$+\dot{\theta}_z + \sin\theta_I\}\{\sin(\theta_G + \phi_k)(\dot{w}'_k + \dot{\theta}_I < \beta_s \cos\theta_I - \beta_d \sin\theta_I - 1 >)$$

$$+\cos(\theta_G + \phi_k)(\dot{v}'_k + \dot{\theta}_I < 1 - \beta_d \cos\theta_I - \beta_s \sin\theta_I > + \cos\theta_I)\}]$$

$$-Im_{\eta\eta}[\{\sin(\theta_G + \phi_k)(\dot{w}'_k + \dot{\theta}_x \sin\psi_k - \dot{\theta}_y \cos\psi_k - \dot{\theta}_I < \beta_s \cos\theta_I - \beta_d \sin\theta_I + 1 >)$$

$$+\cos(\theta_G + \phi_k)(\dot{v}'_k - \dot{\theta}_z - \dot{\theta}_I < 1 - \beta_d \cos\theta_I - \beta_s \sin\theta_I > - \cos\theta_I)\}$$

$$\{\sin(\theta_G + \phi_k)(-\dot{v}'_k$$

$$+\dot{\theta}_I < 1 - \beta_d \cos\theta_I - \beta_s \sin\theta_I > + \cos\theta_I) + \cos(\theta_G + \phi_k)(\dot{w}'_k$$

$$-\dot{\theta}_I < \beta_s \cos\theta_I - \beta_d \sin\theta_I - 1 >)\}]$$

$$-Im_{\eta\zeta}[\{\sin(\theta_G + \phi_k)(\dot{v}'_k + \dot{\theta}_z + \dot{\theta}_I < 1 - \beta_d \cos\theta_I - \beta_s \sin\theta_I > + \cos\theta_I)$$

$$+\cos(\theta_G + \phi_k)(-\dot{w}'_k + \dot{\theta}_x \sin\psi_k \cos\theta_I - \dot{\theta}_y \cos\psi_k \cos\theta_I$$

$$-\dot{\theta}_I < \beta_s \cos\theta_I - \beta_d \sin\theta_I + 1 >)$$

$$+\dot{\theta}_z + \sin\theta_I\}\{\sin(\theta_G + \phi_k)(-\dot{v}'_k + \dot{\theta}_I < 1 - \beta_d \cos\theta_I - \beta_s \sin\theta_I > + \cos\theta_I)$$

$$+\cos(\theta_G + \phi_k)(\dot{w}'_k - \dot{\theta}_I < \beta_s \cos\theta_I - \beta_d \sin\theta_I - 1 >)\}$$

$$+\{\sin(\theta_G + \phi_k)(\dot{w}'_k + \dot{\theta}_I < \beta_s \cos\theta_I - \beta_d \sin\theta_I - 1 >)$$

$$+\cos(\theta_G + \phi_k)(\dot{v}'_k + \dot{\theta}_I < 1 - \beta_d \cos\theta_I - \beta_s \sin\theta_I >)\}$$

$$\{\sin(\theta_G + \phi_k)(\dot{w}'_k + \dot{\theta}_x \sin\psi_k - \dot{\theta}_y \cos\psi_k - \dot{\theta}_I < \beta_s \cos\theta_I - \beta_d \sin\theta_I + 1 >)$$

$$+\cos(\theta_G+\phi_k)(\dot{v}'_k-\dot{\theta}_z-\dot{\theta}_I<1-\beta_d\cos\theta_I-\beta_s\sin\theta_I>-\cos\theta_I)\}]$$

$$-m\eta_m[\{\sin(\theta_G+\phi_k)(\dot{v}'_k+\dot{\theta}_z+\dot{\theta}_I<1-\beta_d\cos\theta_I-\beta_s\sin\theta_I>+\cos\theta_I)$$

$$+\cos(\theta_G+\phi_k)(-\dot{w}'_k+\dot{\theta}_x\sin\psi_k\cos\theta_I-\dot{\theta}_y\cos\psi_k\cos\theta_I$$

$$-\dot{\theta}_I<\beta_s\cos\theta_I-\beta_d\sin\theta_I+1>)$$

$$+\dot{\theta}_z\sin\theta_I)+\sin\theta_I\}\{(\dot{\theta}_I+1)(v_k+a)+\dot{\theta}_I w_k\}]$$

$$-m\zeta_m[\{\sin(\theta_G+\phi_k)(\dot{w}'_k+\dot{\theta}_x\sin\psi_k-\dot{\theta}_y\cos\psi_k-\dot{\theta}_I<\beta_s\cos\theta_I-\beta_d\sin\theta_I+1>)$$

$$+\cos(\theta_G+\phi_k)(\dot{v}'_k-\dot{\theta}_z-\dot{\theta}_I<1-\beta_d\cos\theta_I-\beta_s\sin\theta_I>-\cos\theta_I)\}$$

$$\{(\dot{\theta}_I+1)(v_k+a)+\dot{\theta}_I w_k\}]$$

$$+Im_{\zeta\zeta}[\{\sin(\theta_G+\phi_k)(\dot{v}'_k+\dot{\theta}_I<1-\beta_d\cos\theta_I-\beta_s\sin\theta_I>+1)$$

$$+\cos(\theta_G+\phi_k)(-\dot{w}'_k-\dot{\theta}_I<\beta_s\cos\theta_I-\beta_d\sin\theta_I+1>)\}$$

$$\{\sin(\theta_G+\phi_k)(\dot{\theta}_x\sin\psi_k-\dot{\theta}_y\cos\psi_k)\cos\theta_I-\cos(\theta_G+\phi_k)\dot{\theta}_z\}]$$

$$+Im_{\eta\eta}[\{\sin(\theta_G+\phi_k)(\dot{w}'_k-\dot{\theta}_I<\beta_s\cos\theta_I-\beta_d\sin\theta_I+1>)$$

$$+\cos(\theta_G+\phi_k)(\dot{v}'_k-\dot{\theta}_I<1-\beta_d\cos\theta_I-\beta_s\sin\theta_I>-1)\}$$

$$\{-\sin(\theta_G+\phi_k)\dot{\theta}_z-\cos(\theta_G+\phi_k)(\dot{\theta}_x\sin\psi_k-\dot{\theta}_y\cos\psi_k)\cos\theta_I\}]$$

$$+Im_{\eta\zeta}[\{\sin(\theta_G+\phi_k)(\dot{v}'_k+\dot{\theta}_I<1-\beta_d\cos\theta_I-\beta_s\sin\theta_I>+1)$$

$$+\cos(\theta_G+\phi_k)(-\dot{w}'_k-\dot{\theta}_I<\beta_s\cos\theta_I-\beta_d\sin\theta_I+1>)\}\{-\sin(\theta_G+\phi_k)\dot{\theta}_z$$

$$-\cos(\theta_G+\phi_k)(\dot{\theta}_x\sin\psi_k-\dot{\theta}_y\cos\psi_k)\}$$

$$+\{\sin(\theta_G+\phi_k)(\dot{w}'_k-\dot{\theta}_I<\beta_s\cos\theta_I-\beta_d\sin\theta_I+1>)+\cos(\theta_G+\phi_k)(\dot{v}'_k$$

$$-\dot{\theta}_I<1-\beta_d\cos\theta_I-\beta_s\sin\theta_I>-1)\}\{\sin(\theta_G+\phi_k)(\dot{\theta}_x\sin\psi_k-\dot{\theta}_y\cos\psi_k)\cos\theta_I$$

$$-\cos(\theta_G+\phi_k)\dot{\theta}_z\}]$$

$$+m\eta_m[\{\sin(\theta_G+\phi_k)(\dot{v}'_k+\dot{\theta}_I<1-\beta_d\cos\theta_I-\beta_s\sin\theta_I>+1)+\cos(\theta_G+\phi_k)(-\dot{w}'_k$$

$$-\dot{\theta}_I<\beta_s\cos\theta_I-\beta_d\sin\theta_I+1>)\}\{\dot{R}_x\cos\psi_k+\dot{R}_y\sin\psi_k\}]$$

$$+m\zeta_m[\{\sin(\theta_G+\phi_k)(\dot{w}'_k-\dot{\theta}_I<\beta_s\cos\theta_I-\beta_d\sin\theta_I+1>)+\cos(\theta_G+\phi_k)(\dot{v}'_k$$

$$-\dot{\theta}_I<1-\beta_d\cos\theta_I-\beta_s\sin\theta_I>-1)\}\{\dot{R}_x\cos\psi_k+\dot{R}_y\sin\psi_k\}]$$

$$+Im_{\zeta\zeta}[\{\sin(\theta_G+\phi_k)(1+\dot\theta_I)-\cos(\theta_G+\phi_k)\dot\theta_I\}$$

$$\{\sin(\theta_G+\phi_k)v_k'\dot\phi_k-\cos(\theta_G+\phi_k)w_k'\dot\phi_k+\sin\theta_I\}]$$

$$-Im_{\eta\eta}[\{\sin(\theta_G+\phi_k)\dot\theta_I+\cos(\theta_G+\phi_k)(1+\dot\theta_I)\}$$

$$\{\sin(\theta_G+\phi_k)w_k'\dot\phi_k+\cos(\theta_G+\phi_k)v_k'\dot\phi_k+\sin\theta_I\}]$$

$$+Im_{\eta\zeta}[\{\sin(\theta_G+\phi_k)(1+\dot\theta_I)-\cos(\theta_G+\phi_k)\dot\theta_I\}$$

$$\{\sin(\theta_G+\phi_k)w_k'\dot\phi_k+\cos(\theta_G+\phi_k)v_k'\dot\phi_k-\sin\theta_I\}$$

$$-\{\sin(\theta_G+\phi_k)\dot\theta_I+\cos(\theta_G+\phi_k)(1+\dot\theta_I)\}$$

$$\{\sin(\theta_G+\phi_k)v_k'\dot\phi_k-\cos(\theta_G+\phi_k)w_k'\dot\phi_k+\sin\theta_I\}]$$

$$+m\eta_m[\{\sin(\theta_G+\phi_k)(1+\dot\theta_I)-\cos(\theta_G+\phi_k)\dot\theta_I\}\{-\beta_s\cos\theta_I+\beta_d\sin\theta_I+\dot u_k$$

$$-w_k(\dot\theta_x\sin\psi_k-\dot\theta_y\cos\psi_k)-\dot\theta_z(a+v_k)-(e_1+e_2)(\beta_s\cos\theta_I-\beta_d\sin\theta_I)$$

$$+\dot\theta_I(-w_k+e_1+e_2)(\beta_s\cos\theta_I-\beta_d\sin\theta_I)$$

$$+\dot\theta_I(e_1+e_2-a-v_k)(\beta_d\cos\theta_I+\beta_s\sin\theta_I)\cos\theta_I-w_k\}]$$

$$-m\zeta_m[\{\sin(\theta_G+\phi_k)\dot\theta_I+\cos(\theta_G+\phi_k)(1+\dot\theta_I)\}\{-\beta_s\cos\theta_I+\beta_d\sin\theta_I+\dot u_k$$

$$-w_k(\dot\theta_x\sin\psi_k-\dot\theta_y\cos\psi_k)-\dot\theta_z(a+v_k)-(e_1+e_2)(\beta_s\cos\theta_I-\beta_d\sin\theta_I)$$

$$+\dot\theta_I(-w_k+e_1+e_2)(\beta_s\cos\theta_I-\beta_d\sin\theta_I)$$

$$+\dot\theta_I(e_1+e_2-a-v_k)(\beta_d\cos\theta_I+\beta_s\sin\theta_I)\cos\theta_I-w_k\}]$$

$$+Im_{\zeta\zeta}[\{\sin(\theta_G+\phi_k)(\dot\theta_z v_k'+\dot\theta_I v_k'<\beta_d\cos\theta_I+\beta_s\sin\theta_I>+\dot\theta_I v_k'w_k')$$

$$-\cos(\theta_G+\phi_k)(\dot\theta_z w_k'+\dot\theta_I w_k'<\beta_d\cos\theta_I+\beta_s\sin\theta_I>)\}\{\dot\theta_I\sin(\theta_G+\phi_k)\}]$$

$$+Im_{\eta\eta}[\{\sin(\theta_G+\phi_k)(-\dot\theta_z w_k'+\dot\theta_I w_k'<\beta_d\cos\theta_I+\beta_s\sin\theta_I>)+\cos(\theta_G+\phi_k)(\dot\theta_z v_k'$$

$$+\dot\theta_I v_k'<\beta_d\cos\theta_I+\beta_s\sin\theta_I>+\dot\theta_I v_k'w_k')\}\{-\dot\theta_I\cos(\theta_G+\phi_k)\}]$$

$$+Im_{\eta\zeta}[\{\sin(\theta_G+\phi_k)(\dot\theta_z v_k'+\dot\theta_I v_k'<\beta_d\cos\theta_I+\beta_s\sin\theta_I>+\dot\theta_I v_k'w_k')$$

$$-\cos(\theta_G+\phi_k)(\dot\theta_z w_k'+\dot\theta_I w_k'<\beta_d\cos\theta_I+\beta_s\sin\theta_I>)\}\{-\dot\theta_I\cos(\theta_G+\phi_k)\}$$

$$+\{\sin(\theta_G+\phi_k)(-\dot\theta_z w_k'+\dot\theta_I w_k'<\beta_d\cos\theta_I+\beta_s\sin\theta_I>)+\cos(\theta_G+\phi_k)(\dot\theta_z v_k'$$

$$+\dot\theta_I v_k'<\beta_d\cos\theta_I+\beta_s\sin\theta_I>+\dot\theta_I v_k'w_k')\}\{\dot\theta_I\sin(\theta_G+\phi_k)\}]$$

$$+m\eta_m[\{\sin(\theta_G+\phi_k)(\dot\theta_z v_k' + \dot\theta_I v_k' < \beta_d\cos\theta_I + \beta_s\sin\theta_I > +\dot\theta_I v_k' w_k')$$
$$-\cos(\theta_G+\phi_k)(\dot\theta_z w_k' + \dot\theta_I w_k' < \beta_d\cos\theta_I+\beta_s\sin\theta_I >)\}\{(< n-1 > l_e + x_k)(1+\dot\theta_I)\}]$$
$$+m\zeta_m[\{\sin(\theta_G+\phi_k)(-\dot\theta_z w_k' + \dot\theta_I w_k' < \beta_d\cos\theta_I+\beta_s\sin\theta_I >) + \cos(\theta_G+\phi_k)(\dot\theta_z v_k'$$
$$+\dot\theta_I v_k' < \beta_d\cos\theta_I + \beta_s\sin\theta_I > +\dot\theta_I v_k' w_k')\}\{(< n-1 > l_e + x_k)(1+\dot\theta_I)\}]$$
$$+Im_{\zeta\zeta}[-\sin(\theta_G+\phi_k)v_k' + \cos(\theta_G+\phi_k)\{-\dot\phi_k + (\dot\theta_x\cos\psi_k + \dot\theta_y\sin\psi_k)$$
$$-\dot\theta_I(\beta_d < \cos\theta_I+\sin\theta_I > +\beta_s < \sin\theta_I - \cos\theta_I > +1) + (\beta_p+\beta_d\cos\theta_I+\beta_s\sin\theta_I) + w_k'\}]$$
$$[\sin(\theta_G+\phi_k)\{(-\dot\phi_k + w_k') + \dot\theta_I(\beta_d < \cos\theta_I+\sin\theta_I > +\beta_s < \sin\theta_I-\cos\theta_I > +1)$$
$$+(\beta_p+\beta_d\cos\theta_I+\beta_s\sin\theta_I)\} + \cos(\theta_G+\phi_k)\{v_k'(\dot\theta_I\cos\theta_I - 1)\}]$$
$$+Im_{\eta\eta}[\sin(\theta_G+\phi_k)\{\dot\phi_k + \dot\theta_x\cos\psi_k + \dot\theta_y\sin\psi_k + \dot\theta_I(\beta_d < \cos\theta_I + \sin\theta_I >$$
$$+\beta_s < \sin\theta_I - \cos\theta_I > +1) - (\beta_p + \beta_d\cos\theta_I + \beta_s\sin\theta_I) - w_k'\} - \cos(\theta_G+\phi_k)v_k']$$
$$[\sin(\theta_G+\phi_k)\{-v_k'(\dot\theta_I - 1)\} + \cos(\theta_G+\phi_k)\{-\dot\phi_k + w_k' - \dot\theta_I(\beta_d < \cos\theta_I+\sin\theta_I >$$
$$+\beta_s < \sin\theta_I - \cos\theta_I > +1) + (\beta_p+\beta_d\cos\theta_I+\beta_s\sin\theta_I)\}]$$
$$+Im_{\eta\zeta}[-\sin(\theta_G+\phi_k)v_k' + \cos(\theta_G+\phi_k)\{-\dot\phi_k + (\dot\theta_x\cos\psi_k + \dot\theta_y\sin\psi_k)$$
$$-\dot\theta_I(\beta_d < \cos\theta_I+\sin\theta_I > +\beta_s < \sin\theta_I-\cos\theta_I > +1)$$
$$+(\beta_p+\beta_d\cos\theta_I+\beta_s\sin\theta_I) + w_k'\}]$$
$$[\sin(\theta_G+\phi_k)\{-v_k'(\dot\theta_I - 1)\} + \cos(\theta_G+\phi_k)\{-\dot\phi_k + w_k'$$
$$-\dot\theta_I(\beta_d < \cos\theta_I+\sin\theta_I > +\beta_s < \sin\theta_I-\cos\theta_I > +1) + (\beta_p+\beta_d\cos\theta_I+\beta_s\sin\theta_I)\}]$$
$$+Im_{\eta\zeta}[\sin(\theta_G+\phi_k)\{\dot\phi_k + \dot\theta_x\cos\psi_k + \dot\theta_y\sin\psi_k + +\dot\theta_I(\beta_d < \cos\theta_I+\sin\theta_I >$$
$$+\beta_s < \sin\theta_I-\cos\theta_I > +1) - (\beta_p+\beta_d\cos\theta_I+\beta_s\sin\theta_I) - w_k'\} - \cos(\theta_G+\phi_k)v_k']$$
$$[\sin(\theta_G+\phi_k)\{(-\dot\phi_k + w_k') + \dot\theta_I(\beta_d < \cos\theta_I+\sin\theta_I > +\beta_s < \sin\theta_I-\cos\theta_I > +1)$$
$$+(\beta_p+\beta_d\cos\theta_I+\beta_s\sin\theta_I)\} + \cos(\theta_G+\phi_k)\{v_k'(\dot\theta_I - 1)\}]$$
$$+m\eta_m[-\sin(\theta_G+\phi_k)v_k' + \cos(\theta_G+\phi_k)\{-\dot\phi_k + (\dot\theta_x\cos\psi_k + \dot\theta_y\sin\psi_k)$$
$$-\dot\theta_I(\beta_d < \cos\theta_I+\sin\theta_I > +\beta_s < \sin\theta_I-\cos\theta_I > +1)$$
$$+(\beta_p+\beta_d\cos\theta_I+\beta_s\sin\theta_I) + w_k'\}]$$

$$[-\{\dot{\theta}_I w_k + (e_1 + e_2 + < n-1 > l_e + x_k)(1 - \dot{\theta}_I \cos\theta_I)\}]$$

$$+m\zeta_m[\sin(\theta_G + \phi_k)\{\dot{\phi}_k + \dot{\theta}_x \cos\psi_k + \dot{\theta}_y \sin\psi_k + \dot{\theta}_I(\beta_d < \cos\theta_I + \sin\theta_I >$$

$$+\beta_s < \sin\theta_I - \cos\theta_I > +1) - (\beta_p + \beta_d \cos\theta_I + \beta_s \sin\theta_I) - w_k'\} - \cos(\theta_G + \phi_k)v_k']$$

$$[-\{\dot{\theta}_I w_k + (e_1 + e_2 + < n-1 > l_e + x_k)(1 - \dot{\theta}_I \cos\theta_I)\}]$$

$$+Im_{\zeta\zeta}[-v_k' \sin(\theta_G + \phi_k) - \cos(\theta_G + \phi_k)\{\dot{\phi}_k + \dot{\theta}_I(\beta_d < \cos\theta_I + \sin\theta_I >$$

$$+\beta_s < \sin\theta_I - \cos\theta_I > +1) - (\beta_p + \beta_d \cos\theta_I + \beta_s \sin\theta_I) - w_k'\}]$$

$$[\sin(\theta_G + \phi_k)(\dot{\theta}_x \cos\psi_k + \dot{\theta}_y \sin\psi_k)]$$

$$+Im_{\eta\eta}[\sin(\theta_G + \phi_k)\{\dot{\phi}_k + \dot{\theta}_I(\beta_d < \cos\theta_I + \sin\theta_I > +\beta_s < \sin\theta_I - \cos\theta_I > +1)$$

$$-(\beta_p + \beta_d \cos\theta_I + \beta_s \sin\theta_I) - w_k'\} - \cos(\theta_G + \phi_k)v_k']$$

$$[-\cos(\theta_G + \phi_k)(\dot{\theta}_x \cos\psi_k + \dot{\theta}_y \sin\psi_k)]$$

$$+Im_{\eta\zeta}[-v_k' \sin(\theta_G + \phi_k) - \cos(\theta_G + \phi_k)\{\dot{\phi}_k + \dot{\theta}_I(\beta_d < \cos\theta_I + \sin\theta_I >$$

$$+\beta_s < \sin\theta_I - \cos\theta_I > +1) - (\beta_p + \beta_d \cos\theta_I + \beta_s \sin\theta_I) - w_k'\}]$$

$$[-\cos(\theta_G + \phi_k)(\dot{\theta}_x \cos\psi_k + \dot{\theta}_y \sin\psi_k)]$$

$$+Im_{\eta\zeta}[\sin(\theta_G + \phi_k)\{\dot{\phi}_k + \dot{\theta}_I(\beta_d < \cos\theta_I + \sin\theta_I > +\beta_s < \sin\theta_I - \cos\theta_I > +1)$$

$$-(\beta_p + \beta_d \cos\theta_I + \beta_s \sin\theta_I) - w_k'\} - \cos(\theta_G + \phi_k)v_k'][\sin(\theta_G + \phi_k)(\dot{\theta}_x \cos\psi_k + \dot{\theta}_y \sin\psi_k)]$$

$$+m\eta_m[-v_k' \sin(\theta_G + \phi_k) - \cos(\theta_G + \phi_k)\{\dot{\phi}_k + \dot{\theta}_I(\beta_d < \cos\theta_I + \sin\theta_I >$$

$$+\beta_s < \sin\theta_I - \cos\theta_I > +1) - (\beta_p + \beta_d \cos\theta_I + \beta_s \sin\theta_I) - w_k'\}][-\dot{R}_x \sin\psi_k \cos\theta_I$$

$$+\dot{R}_y \cos\psi_k \cos\theta_I + (\dot{\theta}_z + \beta_d \cos\theta_I + \beta_s \sin\theta_I)(< n-1 > l_e + x_k)]$$

$$+m\zeta_m[\sin(\theta_G + \phi_k)\{\dot{\phi}_k + \dot{\theta}_I(\beta_d < \cos\theta_I + \sin\theta_I > +\beta_s < \sin\theta_I - \cos\theta_I > +1)$$

$$-(\beta_p + \beta_d \cos\theta_I + \beta_s \sin\theta_I) - w_k'\} - \cos(\theta_G + \phi_k)v_k']$$

$$[-\dot{R}_x \sin\psi_k \cos\theta_I + \dot{R}_y \cos\psi_k \cos\theta_I$$

$$+(\dot{\theta}_z + \beta_d \cos\theta_I + \beta_s \sin\theta_I)(< n-1 > l_e + x_k)]$$

$$+Im_{\zeta\zeta}[-\dot{\theta}_I \cos(\theta_G + \phi_k)][\sin(\theta_G + \phi_k)\{\dot{\theta}_z w_k' \cos\theta_I + \dot{\theta}_I w_k'(\beta_d \cos\theta_I + \beta_s \sin\theta_I)\}$$

$$+\cos(\theta_G + \phi_k)\{-\dot{\theta}_z v_k' \cos\theta_I + \dot{\theta}_I v_k'(\beta_d \cos\theta_I + \beta_s \sin\theta_I) - \dot{\theta}_I v_k' w_k'\}]$$

$+Im_{\eta\eta}[\dot{\theta}_I \sin(\theta_G + \phi_k)][\sin(\theta_G + \phi_k)\{\dot{\theta}_z v_k' \cos\theta_I - \dot{\theta}_I v_k'(\beta_d \cos\theta_I + \beta_s \sin\theta_I)$

$+\dot{\theta}_I v_k' w_k'\} + \cos(\theta_G + \phi_k)\{\dot{\theta}_z w_k' \cos\theta_I + \dot{\theta}_I w_k'(\beta_d \cos\theta_I + \beta_s \sin\theta_I)\}]$

$+Im_{\eta\zeta}[-\dot{\theta}_I \cos(\theta_G + \phi_k)][\sin(\theta_G + \phi_k)\{\dot{\theta}_z v_k' \cos\theta_I - \dot{\theta}_I v_k'(\beta_d \cos\theta_I + \beta_s \sin\theta_I)$

$+\dot{\theta}_I v_k' w_k'\} + \cos(\theta_G + \phi_k)\{\dot{\theta}_z w_k' \cos\theta_I + \dot{\theta}_I w_k'(\beta_d \cos\theta_I + \beta_s \sin\theta_I)\}]$

$+Im_{\eta\zeta}[\dot{\theta}_I \sin(\theta_G + \phi_k)][\sin(\theta_G + \phi_k)\{\dot{\theta}_z w_k' \cos\theta_I + \dot{\theta}_I w_k'(\beta_d \cos\theta_I + \beta_s \sin\theta_I)\}$

$+\cos(\theta_G + \phi_k)\{-\dot{\theta}_z v_k' \cos\theta_I + \dot{\theta}_I v_k'(\beta_d \cos\theta_I + \beta_s \sin\theta_I) - \dot{\theta}_I v_k' w_k'\}]$

$+m\eta_m[-\dot{\theta}_I \cos(\theta_G + \phi_k)][(\dot{R}_x \cos\psi_k + \dot{R}_y \sin\psi_k)(\beta_s \cos\theta_I - \beta_d \sin\theta_I)$

$-\dot{\theta}_I(\beta_s \sin\theta_I + \beta_d \cos\theta_I)(e_1 + e_2 + 2a \sin\theta_I) - w_k(\dot{\theta}_x \cos\psi_k + \dot{\theta}_y \sin\psi_k)\dot{\theta}_z \cos\theta_I(e_1 + e_2)$

$+\dot{\theta}_I(a \cos\theta_I < \beta_s \cos\theta_I - \beta_d \sin\theta_I > -u_k) - (\beta_p + \beta_d \cos\theta_I + \beta_s \sin\theta_I)w_k$

$+a \cos\theta_I(\beta_s \cos\theta_I - \beta_d \sin\theta_I)]$

$+m\zeta_m[\dot{\theta}_I \sin(\theta_G + \phi_k)][(\dot{R}_x \cos\psi_k + \dot{R}_y \sin\psi_k)(\beta_s \cos\theta_I - \beta_d \sin\theta_I)$

$-\dot{\theta}_I(\beta_s \sin\theta_I + \beta_d \cos\theta_I)(e_1 + e_2 + 2a \sin\theta_I) - w_k(\dot{\theta}_x \cos\psi_k + \dot{\theta}_y \sin\psi_k)\dot{\theta}_z \cos\theta_I(e_1 + e_2)$

$+\dot{\theta}_I(a \cos\theta_I < \beta_s \cos\theta_I - \beta_d \sin\theta_I > -u_k) - (\beta_p + \beta_d \cos\theta_I + \beta_s \sin\theta_I)w_k$

$+a \cos\theta_I(\beta_s \cos\theta_I - \beta_d \sin\theta_I)]$

$+Im_{\zeta\zeta}[\sin(\theta_G + \phi_k)\{v_k'(\cos\theta_I + \sin\theta_I)(\dot{\theta}_x \sin\psi_k - \dot{\theta}_y \cos\psi_k$

$+\dot{\theta}_I < \beta_s \cos\theta_I - \beta_d \sin\theta_I >)\}$

$+\cos(\theta_G + \phi_k)\{-w_k'(\dot{\theta}_x \sin\psi_k - \dot{\theta}_y \cos\psi_k + \dot{\theta}_I < \beta_s \cos\theta_I - \beta_d \sin\theta_I >)\}]$

$[\dot{\theta}_I \cos(\theta_G + \phi_k)]$

$+Im_{\eta\eta}[\{w_k' \sin(\theta_G + \phi_k) + v_k' \cos(\theta_G + \phi_k)\}(\cos\theta_I + \sin\theta_I)\{\dot{\theta}_x \sin\psi_k - \dot{\theta}_y \cos\psi_k$

$+\dot{\theta}_I(\beta_s \cos\theta_I - \beta_d \sin\theta_I)\}][\dot{\theta}_I \sin(\theta_G + \phi_k)]$

$+Im_{\eta\zeta}[\sin(\theta_G + \phi_k)\{v_k'(\cos\theta_I + \sin\theta_I)(\dot{\theta}_x \sin\psi_k - \dot{\theta}_y \cos\psi_k$

$+\dot{\theta}_I < \beta_s \cos\theta_I - \beta_d \sin\theta_I >)\}$

$+\cos(\theta_G + \phi_k)\{-w_k'(\dot{\theta}_x \sin\psi_k - \dot{\theta}_y \cos\psi_k + \dot{\theta}_I < \beta_s \cos\theta_I - \beta_d \sin\theta_I >)\}][\dot{\theta}_I \sin(\theta_G + \phi$

$+Im_{\eta\zeta}[\{w_k' \sin(\theta_G + \phi_k) + v_k' \cos(\theta_G + \phi_k)\}(\cos\theta_I + \sin\theta_I)\{\dot{\theta}_x \sin\psi_k - \dot{\theta}_y \cos\psi_k$

$$+\dot\theta_I(\beta_s\cos\theta_I-\beta_d\sin\theta_I)\}][\dot\theta_I\cos(\theta_G+\phi_k)]$$

$$+m\eta_m[\sin(\theta_G+\phi_k)(\cos\theta_I+\sin\theta_I)\{v_k'(\dot\theta_x\sin\psi_k-\dot\theta_y\cos\psi_k+\dot\theta_I<\beta_s\cos\theta_I-\beta_d\sin\theta_I>)\}$$

$$+\cos(\theta_G+\phi_k)(\cos\theta_I+\sin\theta_I)\{-w_k'(\dot\theta_x\sin\psi_k-\dot\theta_y\cos\psi_k+\dot\theta_I<\beta_s\cos\theta_I-\beta_d\sin\theta_I>)\}]$$

$$[<n-1>l_e+x_k]$$

$$+m\zeta_m[\{w_k'\sin(\theta_G+\phi_k)+v_k'\cos(\theta_G+\phi_k)\}(\cos\theta_I+\sin\theta_I)\{\dot\theta_x\sin\psi_k-\dot\theta_y\cos\psi_k$$

$$+\dot\theta_I(\beta_s\cos\theta_I-\beta_d\sin\theta_I)\}][<n-1>l_e+x_k]$$

$$+Im_{\zeta\zeta}[\sin(\theta_G+\phi_k)\{\dot\phi_k-\dot\theta_x\cos\psi_k-\dot\theta_y\sin\psi_k-\dot\theta_I(\beta_d<\cos\theta_I+\sin\theta_I>$$

$$+\beta_s<\sin\theta_I-\cos\theta_I>+1+v_k')-(\beta_p+\beta_d\cos\theta_I+\beta_s\sin\theta_I)\}-\dot\theta_I w_k'\cos(\theta_G+\phi_k)]$$

$$[\sin(\theta_G+\phi_k)\dot\theta_I w_k'+\cos(\theta_G+\phi_k)\{-\dot\phi_k+\dot\theta_I(\beta_d<\cos\theta_I+\sin\theta_I>$$

$$+\beta_s<\sin\theta_I-\cos\theta_I>+1)+(\beta_p+\beta_d\cos\theta_I+\beta_s\sin\theta_I)+\dot\theta_I v_k'\}]$$

$$+Im_{\eta\eta}[\dot\theta_I w_k'\sin(\theta_G+\phi_k)+\cos(\theta_G+\phi_k)\{-\dot\phi_k+\dot\theta_x\cos\psi_k+\dot\theta_y\sin\psi_k$$

$$+\dot\theta_I(\beta_d<\cos\theta_I+\sin\theta_I>+\beta_s<\sin\theta_I-\cos\theta_I>+1)+\dot\theta_I v_k'$$

$$+(\beta_p+\beta_d\cos\theta_I+\beta_s\sin\theta_I)\}][\sin(\theta_G+\phi_k)\{-\dot\phi_k+\dot\theta_I(\beta_d<\cos\theta_I+\sin\theta_I>$$

$$+\beta_s<\sin\theta_I-\cos\theta_I>+1)+(\beta_p+\beta_d\cos\theta_I+\beta_s\sin\theta_I)-\dot\theta_I v_k'\}+\dot\theta_I w_k'\cos(\theta_G+\phi_k)]$$

$$+Im_{\eta\zeta}[\sin(\theta_G+\phi_k)\{\dot\phi_k-\dot\theta_x\cos\psi_k-\dot\theta_y\sin\psi_k-\dot\theta_I(\beta_d<\cos\theta_I+\sin\theta_I>$$

$$+\beta_s<\sin\theta_I-\cos\theta_I>+1+v_k')-(\beta_p+\beta_d\cos\theta_I+\beta_s\sin\theta_I)\}-\dot\theta_I w_k'\cos(\theta_G+\phi_k)]$$

$$[\sin(\theta_G+\phi_k)\{-\dot\phi_k+\dot\theta_I(\beta_d<\cos\theta_I+\sin\theta_I>+\beta_s<\sin\theta_I-\cos\theta_I>+1)$$

$$+(\beta_p+\beta_d\cos\theta_I+\beta_s\sin\theta_I)-\dot\theta_I v_k'\}+\dot\theta_I w_k'\cos(\theta_G+\phi_k)]$$

$$+Im_{\eta\zeta}[\sin(\theta_G+\phi_k)\dot\theta_I w_k'+\cos(\theta_G+\phi_k)\{-\dot\phi_k+\dot\theta_I(\beta_d<\cos\theta_I+\sin\theta_I>$$

$$+\beta_s<\sin\theta_I-\cos\theta_I>+1)+(\beta_p+\beta_d\cos\theta_I+\beta_s\sin\theta_I)+\dot\theta_I v_k'\}][\dot\theta_I w_k'\sin(\theta_G+\phi_k)$$

$$+\cos(\theta_G+\phi_k)\{-\dot\phi_k+\dot\theta_x\cos\psi_k+\dot\theta_y\sin\psi_k+\dot\theta_I(\beta_d<\cos\theta_I+\sin\theta_I>$$

$$+\beta_s<\sin\theta_I-\cos\theta_I>+1)+\dot\theta_I v_k'+(\beta_p+\beta_d\cos\theta_I+\beta_s\sin\theta_I)\}]$$

$$+m\eta_m[\sin(\theta_G+\phi_k)\{\dot\phi_k-\dot\theta_x\cos\psi_k-\dot\theta_y\sin\psi_k-\dot\theta_I(\beta_d<\cos\theta_I+\sin\theta_I>$$

$$+\beta_s<\sin\theta_I-\cos\theta_I>+1+v_k')-(\beta_p+\beta_d\cos\theta_I+\beta_s\sin\theta_I)\}-\dot\theta_I w_k'\cos(\theta_G+\phi_k)]$$

$$[\dot{w}_k + a\cos\theta_I + v_k - (\beta_s \cos\theta_I - \beta_d \sin\theta_I + 1)(< n-1 > l_e + x_k) - (e_1 + e_2)]$$

$$+m\zeta_m[\dot{\theta}_I w_k' \sin(\theta_G + \phi_k) + \cos(\theta_G + \phi_k)\{-\dot{\phi}_k + \dot{\theta}_x \cos\psi_k + \dot{\theta}_y \sin\psi_k$$

$$+\dot{\theta}_I(\beta_d < \cos\theta_I + \sin\theta_I > +\beta_s < \sin\theta_I - \cos\theta_I > +1) + \dot{\theta}_I v_k'$$

$$+(\beta_p + \beta_d \cos\theta_I + \beta_s \sin\theta_I)\}][\dot{w}_k + a\cos\theta_I + v_k - (\beta_s\cos\theta_I - \beta_d \sin\theta_I + 1)$$

$$(< n-1 > l_e + x_k) - (e_1 + e_2)]$$

$$+Im_{\zeta\zeta}[\sin(\theta_G + \phi_k)\{\dot{\phi}_k - \dot{\theta}_I(\beta_d < \cos\theta_I + \sin\theta_I > +\beta_s < \sin\theta_I - \cos\theta_I > +1)$$

$$+\dot{\theta}_I v_k' - (\beta_p + \beta_d \cos\theta_I + \beta_s \sin\theta_I)\} - \cos(\theta_G + \phi_k)\dot{\theta}_I w_k']$$

$$[\cos(\theta_G + \phi_k)\cos\theta_I(\dot{\theta}_x \cos\psi_k + \dot{\theta}_y \sin\psi_k)]$$

$$+Im_{\eta\eta}[\sin(\theta_G + \phi_k)\dot{\theta}_I w_k' + \cos(\theta_G + \phi_k)\{-\dot{\phi}_k + \dot{\theta}_I(\beta_d < \cos\theta_I + \sin\theta_I >$$

$$+\beta_s < \sin\theta_I - \cos\theta_I > +1) + \dot{\theta}_I v_k' + (\beta_p + \beta_d\cos\theta_I + \beta_s \sin\theta_I)\}]$$

$$[\sin(\theta_G + \phi_k)\cos\theta_I(\dot{\theta}_x \cos\psi_k + \dot{\theta}_y \sin\psi_k)]$$

$$+Im_{\eta\zeta}[\sin(\theta_G + \phi_k)\{\dot{\phi}_k - \dot{\theta}_I(\beta_d < \cos\theta_I + \sin\theta_I > +\beta_s < \sin\theta_I - \cos\theta_I > +1)$$

$$+\dot{\theta}_I v_k' - (\beta_p + \beta_d \cos\theta_I + \beta_s \sin\theta_I)\} - \cos(\theta_G + \phi_k)\dot{\theta}_I w_k']$$

$$[\sin(\theta_G + \phi_k)(\dot{\theta}_x \cos\psi_k + \dot{\theta}_y \sin\psi_k)]$$

$$+Im_{\eta\zeta}[\sin(\theta_G + \phi_k)\dot{\theta}_I w_k' + \cos(\theta_G + \phi_k)\{-\dot{\phi}_k + \dot{\theta}_I(\beta_d < \cos\theta_I + \sin\theta_I >$$

$$+\beta_s < \sin\theta_I - \cos\theta_I > +1) + \dot{\theta}_I v_k' + (\beta_p + \beta_d\cos\theta_I + \beta_s \sin\theta_I)\}]$$

$$[\cos(\theta_G + \phi_k)(\dot{\theta}_x \cos\psi_k + \dot{\theta}_y \sin\psi_k)]$$

$$+m\eta_m[\sin(\theta_G + \phi_k)\{\dot{\phi}_k - \dot{\theta}_I(\beta_d < \cos\theta_I + \sin\theta_I > +\beta_s < \sin\theta_I - \cos\theta_I > +1)$$

$$+\dot{\theta}_I v_k' - (\beta_p + \beta_d \cos\theta_I + \beta_s \sin\theta_I)\} - \cos(\theta_G + \phi_k)\dot{\theta}_I w_k']$$

$$[\dot{R}_z \cos\theta_I + (\dot{\theta}_x \sin\psi_k - \dot{\theta}_y \cos\psi_k)(< n-1 > l_e + x_k)]$$

$$+m\zeta_m[\sin(\theta_G + \phi_k)\dot{\theta}_I w_k' + \cos(\theta_G + \phi_k)\{-\dot{\phi}_k + \dot{\theta}_I(\beta_d < \cos\theta_I + \sin\theta_I >$$

$$+\beta_s < \sin\theta_I - \cos\theta_I > +1) + \dot{\theta}_I v_k' + (\beta_p + \beta_d\cos\theta_I + \beta_s \sin\theta_I)\}]$$

$$[\dot{R}_z \cos\theta_I + (\dot{\theta}_x \sin\psi_k - \dot{\theta}_y \cos\psi_k)(< n-1 > l_e + x_k)]$$

$$+Im_{\zeta\zeta}[-\dot{\theta}_I \sin(\theta_G + \phi_k)][\sin(\theta_G + \phi_k)\{-w_k'(\dot{\theta}_x \sin\psi_k - \dot{\theta}_y \cos\psi_k)$$

$+w'_k(\beta_s \cos\theta_I - \beta_d \sin\theta_I)\dot{\theta}_I\}$

$+\cos(\theta_G+\phi_k)\{-v'_k(\dot{\theta}_x \sin\psi_k - \dot{\theta}_y \cos\psi_k) + v'_k(\beta_s \cos\theta_I - \beta_d \sin\theta_I)\dot{\theta}_I\}]$

$+Im_{\eta\eta}[\dot{\theta}_I \cos(\theta_G+\phi_k)][\sin(\theta_G+\phi_k)\{v'_k(\dot{\theta}_x \sin\psi_k - \dot{\theta}_y \cos\psi_k)$

$-v'_k(\beta_s \cos\theta_I - \beta_d \sin\theta_I)\dot{\theta}_I\}$

$+\cos(\theta_G+\phi_k)\{-w'_k(\dot{\theta}_x \sin\psi_k - \dot{\theta}_y \cos\psi_k) + w'_k(\beta_s \cos\theta_I - \beta_d \sin\theta_I)\dot{\theta}_I\}]$

$+Im_{\eta\zeta}[-\dot{\theta}_I \sin(\theta_G+\phi_k)][\sin(\theta_G+\phi_k)\{v'_k(\dot{\theta}_x \sin\psi_k - \dot{\theta}_y \cos\psi_k)$

$-v'_k(\beta_s \cos\theta_I - \beta_d \sin\theta_I)\dot{\theta}_I\}$

$+\cos(\theta_G+\phi_k)\{-w'_k(\dot{\theta}_x \sin\psi_k - \dot{\theta}_y \cos\psi_k) + w'_k(\beta_s \cos\theta_I - \beta_d \sin\theta_I)\dot{\theta}_I\}]$

$+Im_{\eta\zeta}[\dot{\theta}_I \cos(\theta_G+\phi_k)][\sin(\theta_G+\phi_k)\{-w'_k(\dot{\theta}_x \sin\psi_k - \dot{\theta}_y \cos\psi_k)$

$+w'_k(\beta_s \cos\theta_I - \beta_d \sin\theta_I)\dot{\theta}_I\}$

$+\cos(\theta_G+\phi_k)\{-v'_k(\dot{\theta}_x \sin\psi_k - \dot{\theta}_y \cos\psi_k) + v'_k(\beta_s \cos\theta_I - \beta_d \sin\theta_I)\dot{\theta}_I\}]$

$+[-m\eta_m\dot{\theta}_I \sin(\theta_G+\phi_k) + m\zeta_m\dot{\theta}_I \cos(\theta_G+\phi_k)][-(\dot{R}_x \cos\psi_k + \dot{R}_y \sin\psi_k)$

$(\beta_p + \beta_d \cos\theta_I + \beta_s \sin\theta_I) + \dot{\theta}_I(\beta_d \cos\theta_I - \beta_s \sin\theta_I)(e_1+e_2)$

$+\dot{\theta}_x \cos\psi_k + \dot{\theta}_y \sin\psi_k)(a\cos\theta_I + v_k) + (\dot{\theta}_x \sin\psi_k - \dot{\theta}_y \cos\psi_k)(e_1+e_2) + u_k$

$+a(\beta_s \cos\theta_I - \beta_d \sin\theta_I)\} + a(\beta_p + \beta_d \cos\theta_I + \beta_s \sin\theta_I)]$

$+Im_{\zeta\zeta}[\{(-\dot{v'}_k - w'_k\dot{\phi}_k)\sin(\theta_G+\phi_k) + (-v'_k\dot{\phi}_k + \dot{w}'_k)\cos(\theta_G+\phi_k)\}$

$\{\sin(\theta_G+\phi_k)(\dot{w}'_k + \dot{\theta}_x \sin\psi_k \cos\theta_I - \dot{\theta}_y \cos\psi_k \cos\theta_I$

$-\dot{\theta}_I < \beta_s \cos\theta_I - \beta_d \sin\theta_I - 1 > + \sin\theta_I)$

$+\cos(\theta_G+\phi_k)(-\dot{v'}_k - \dot{\theta}_z - \dot{\theta}_I < 1 - \beta_d \cos\theta_I + \beta_s \sin\theta_I > - \cos\theta_I\}]$

$+Im_{\eta\eta}[\{(v'_k\dot{\phi}_k - \dot{w}'_k)\sin(\theta_G+\phi_k) - (\dot{v}'_k + w'_k\dot{\phi}_k)\cos(\theta_G+\phi_k)\}$

$\{\sin(\theta_G+\phi_k)(\dot{v'}_k - \dot{\theta}_z - \dot{\theta}_I < 1 - \beta_d \cos\theta_I - \beta_s \sin\theta_I > - \cos\theta_I)$

$+\cos(\theta_G+\phi_k)(-\dot{w}'_k + \dot{\theta}_x \sin\psi_k - \dot{\theta}_y \cos\psi_k + \dot{\theta}_I < \beta_s \cos\theta_I - \beta_d \sin\theta_I + 1 > - \sin\theta_I)\}]$

$+Im_{\eta\zeta}[\{(-\dot{v'}_k - w'_k\dot{\phi}_k)\sin(\theta_G+\phi_k) + (-v'_k\dot{\phi}_k + \dot{w}'_k)\cos(\theta_G+\phi_k)\}$

$\{\sin(\theta_G+\phi_k)(\dot{v'}_k - \dot{\theta}_z - \dot{\theta}_I < 1 - \beta_d \cos\theta_I - \beta_s \sin\theta_I > - \cos\theta_I)$

$$+\cos(\theta_G+\phi_k)(-\dot{w}'_k+\dot{\theta}_x\sin\psi_k-\dot{\theta}_y\cos\psi_k+\dot{\theta}_I<\beta_s\cos\theta_I-\beta_d\sin\theta_I+1>-\sin\theta_I)\}]$$

$$+\{(v'_k\dot{\phi}_k-\dot{w}'_k)\sin(\theta_G+\phi_k)-(\dot{v}'_k+w'_k\dot{\phi}_k)\cos(\theta_G+\phi_k)\}$$

$$\{\sin(\theta_G+\phi_k)(\dot{w}'_k+\dot{\theta}_x\sin\psi_k\cos\theta_I-\dot{\theta}_y\cos\psi_k\cos\theta_I$$

$$-\dot{\theta}_I<\beta_s\cos\theta_I-\beta_d\sin\theta_I-1>+\sin\theta_I)$$

$$+\cos(\theta_G+\phi_k)(-\dot{v}'_k-\dot{\theta}_z-\dot{\theta}_I<1-\beta_d\cos\theta_I+\beta_s\sin\theta_I>-\cos\theta_I\}]$$

$$+m\eta_m[\{(-\dot{v}'_k-w'_k\dot{\phi}_k)\sin(\theta_G+\phi_k)+(-v'_k\dot{\phi}_k+\dot{w}'_k)\cos(\theta_G+\phi_k)\}$$

$$\{\dot{R}_x\cos\psi_k+\dot{R}_y\sin\psi_k-\cos\theta_I(a\cos\theta_I+v_k)(1+\dot{\theta}_I)-\sin\theta_I w_k\}]$$

$$+m\zeta_m[\{(v'_k\dot{\phi}_k-\dot{w}'_k)\sin(\theta_G+\phi_k)-(\dot{v}'_k+w'_k\dot{\phi}_k)\cos(\theta_G+\phi_k)\}$$

$$\{\dot{R}_x\cos\psi_k+\dot{R}_y\sin\psi_k-\cos\theta_I(a\cos\theta_I+v_k)(1+\dot{\theta}_I)-\sin\theta_I w_k\}]$$

$$+Im_{\zeta\zeta}[\{v'_k\sin(\theta_G+\phi_k)+w'_k\cos\psi_k\}\{\sin(\theta_G+\phi_k)(\ddot{w}'_k+<\ddot{\theta}_x+\dot{\theta}_y>\sin\psi_k\cos\theta_I$$

$$+<\dot{\theta}_x-\ddot{\theta}_y\dot{>}\cos\psi_k\cos\theta_I-\ddot{\theta}_I<\beta_s\cos\theta_I-\beta_d\sin\theta_I-1>$$

$$+\dot{\theta}_I^2<\beta_s\sin\theta_I+\beta_d\cos\theta_I>$$

$$+\dot{\phi}_k<\dot{v}'_k+\dot{\theta}_z+\cos\theta_I>+\dot{\theta}_I\dot{\phi}_k<1-\beta_d\cos\theta_I+\beta_s\sin\theta_I>)$$

$$+\cos(\theta_G+\phi_k)(\dot{\phi}_k<\dot{w}'_k+\dot{\theta}_x\sin\psi_k-\dot{\theta}_y\cos\psi_k>-\dot{\theta}_I\dot{\phi}_k<\beta_s\cos\theta_I-\beta_d\sin\theta_I-1>$$

$$-\ddot{v}'_k-\ddot{\theta}_z-\ddot{\theta}_I<1-\beta_d\cos\theta_I+\beta_s\sin\theta_I>-\dot{\theta}_I^2<\beta_d\sin\theta_I+\beta_s\cos\theta_I>)\}]$$

$$-Im_{\eta\eta}[\{v'_k\cos(\theta_G+\phi_k)+w'_k\sin\psi_k\}\{\sin(\theta_G+\phi_k)(\ddot{v}'_k-\ddot{\theta}_z$$

$$-\ddot{\theta}_I<1-\beta_d\cos\theta_I+\beta_s\sin\theta_I>-\dot{\theta}_I^2<\beta_d\sin\theta_I-\beta_s\cos\theta_I>$$

$$-\dot{\phi}_k<-\dot{w}'_k-\sin\theta_I)+\dot{\theta}_x\sin\psi_k-\dot{\theta}_y\cos\psi_k>-\dot{\theta}_I\dot{\phi}_k<\beta_s\cos\theta_I-\beta_d\sin\theta_I+1>)$$

$$+\cos(\theta_G+\phi_k)(\dot{\phi}_k<\dot{v}'_k-\dot{\theta}_z-\cos\theta_I>-\dot{\theta}_I\dot{\phi}_k<1-\beta_d\cos\theta_I-\beta_s\sin\theta_I>-\ddot{w}'_k$$

$$+<\ddot{\theta}_x+\dot{\theta}_y>\sin\psi_k\cos\theta_I+<\dot{\theta}_x-\ddot{\theta}_y>\cos\psi_k\cos\theta_I$$

$$+\ddot{\theta}_I<\beta_s\cos\theta_I-\beta_d\sin\theta_I+1>$$

$$+\dot{\theta}_I^2<-\beta_s\sin\theta_I-\beta_d\cos\theta_I>)\}]$$

$$+Im_{\eta\zeta}[\{v'_k\sin(\theta_G+\phi_k)+w'_k\cos\psi_k\}\{\sin(\theta_G+\phi_k)(\ddot{v}'_k-\ddot{\theta}_z$$

$$-\ddot{\theta}_I<1-\beta_d\cos\theta_I+\beta_s\sin\theta_I>-\dot{\theta}_I^2<\beta_d\sin\theta_I-\beta_s\cos\theta_I>$$

$$-\dot{\phi}_k < -\dot{w}'_k - \sin\theta_I) + \dot{\theta}_x \sin\psi_k - \dot{\theta}_y \cos\psi_k > -\dot{\theta}_I \dot{\phi}_k < \beta_s \cos\theta_I - \beta_d \sin\theta_I + 1 >)$$
$$+\cos(\theta_G + \phi_k)(\dot{\phi}_k < \dot{v}'_k - \dot{\theta}_z - \cos\theta_I > -\dot{\theta}_I \dot{\phi}_k < 1 - \beta_d \cos\theta_I - \beta_s \sin\theta_I > -\ddot{w}'_k$$
$$+ < \ddot{\theta}_x + \dot{\theta}_y > \sin\psi_k \cos\theta_I + < \dot{\theta}_x - \ddot{\theta}_y > \cos\psi_k \cos\theta_I + \ddot{\theta}_I < \beta_s \cos\theta_I - \beta_d \sin\theta_I + 1 >$$
$$+\dot{\theta}_I^2 < -\beta_s \sin\theta_I - \beta_d \cos\theta_I >)\}]$$
$$-Im_{\eta\zeta}[\{v'_k \cos(\theta_G + \phi_k) + w'_k \sin\psi_k\}\{\sin(\theta_G + \phi_k)(\ddot{w}'_k + < \ddot{\theta}_x + \dot{\theta}_y > \sin\psi_k \cos\theta_I$$
$$+ < \dot{\theta}_x - \ddot{\theta}_y > \cos\psi_k \cos\theta_I - \ddot{\theta}_I < \beta_s \cos\theta_I - \beta_d \sin\theta_I - 1 >$$
$$+\dot{\theta}_I^2 < \beta_s \sin\theta_I + \beta_d \cos\theta_I >$$
$$+\dot{\phi}_k < \dot{v}'_k + \dot{\theta}_z + \cos\theta_I > +\dot{\theta}_I \dot{\phi}_k < 1 - \beta_d \cos\theta_I + \beta_s \sin\theta_I >)$$
$$+\cos(\theta_G + \phi_k)(\dot{\phi}_k < \dot{w}'_k + \dot{\theta}_x \sin\psi_k - \dot{\theta}_y \cos\psi_k > -\dot{\theta}_I \dot{\phi}_k < \beta_s \cos\theta_I - \beta_d \sin\theta_I - 1 >$$
$$-\ddot{v}'_k - \ddot{\theta}_z - \ddot{\theta}_I < 1 - \beta_d \cos\theta_I + \beta_s \sin\theta_I > -\dot{\theta}_I^2 < \beta_d \sin\theta_I + \beta_s \cos\theta_I >)\}]$$
$$+m\eta_m[\{v'_k \sin(\theta_G + \phi_k) + w'_k \cos\psi_k\}\{(-\dot{R}_x + \ddot{R}_y) \sin\psi_k + (\ddot{R}_x + \dot{R}_y) \cos\psi_k$$
$$-\dot{v}_k(\cos\theta_I + \dot{\theta}_I - (a + v_k)\ddot{\theta}_I - \dot{w}_k \sin\theta_I\}]$$
$$-m\zeta_m[\{v'_k \cos(\theta_G + \phi_k) + w'_k \sin\psi_k\}\{(-\dot{R}_x + \ddot{R}_y) \sin\psi_k + (\ddot{R}_x + \dot{R}_y) \cos\psi_k$$
$$-\dot{v}_k(\cos\theta_I + \dot{\theta}_I - (a + v_k)\ddot{\theta}_I - \dot{w}_k \sin\theta_I\}]$$
$$+Im_{\zeta\zeta}[\{\dot{\phi}_k \cos(\theta_G + \phi_k)\}\{\sin(\theta_G + \phi_k)(-\dot{\phi}_k$$
$$+\dot{\theta}_x \cos\psi_k + \dot{\theta}_y \sin\psi_k - \dot{\theta}_z w'_k \cos\theta_I + \dot{\theta}_I \beta_d \cos\theta_I$$
$$+\sin\theta_I > +\dot{\theta}_I \beta_s < \sin\theta_I - \cos\theta_I > +\dot{\theta}_I + \dot{\theta}_I w'_k < \beta_d \cos\theta_I + \beta_s \sin\theta_I > +\dot{\theta}_I w'_k$$
$$+ < \beta_p + \beta_d \cos\theta_I + \beta_s \sin\theta_I > +w'_k \cos\theta_I) + \cos(\theta_G + \phi_k)(-\dot{\theta}_z v'_k \cos\theta_I$$
$$+v'_k \dot{\theta}_I < \beta_d \cos\theta_I + \beta_s \sin\theta_I > +\dot{\theta}_I v'_k + \dot{\theta}_I w'_k v'_k + v'_k \cos\theta_I)\}]$$
$$-Im_{\eta\eta}[\{\dot{\phi}_k \sin(\theta_G + \phi_k)\}\{\sin(\theta_G + \phi_k)(\dot{\theta}_z v'_k \cos\theta_I - \dot{\theta}_I v'_k < \beta_d \cos\theta_I + \beta_s \sin\theta_I >$$
$$-v'_k \dot{\theta}_I - \dot{\theta}_I v'_k w'_k - v'_k \cos\theta_I) + \cos(\theta_G + \phi_k)(-\dot{\phi}_k - \dot{\theta}_x \cos\psi_k - \dot{\theta}_y \sin\psi_k + \dot{\theta}_z w'_k \cos\theta_I$$
$$-\dot{\theta}_I \beta_d < \cos\theta_I + \sin\theta_I > -\dot{\theta}_I \beta_s < \sin\theta_I - \cos\theta_I > -\dot{\theta}_I$$
$$+\dot{\theta}_I w'_k < \beta_d \cos\theta_I + \beta_s \sin\theta_I >$$
$$+\dot{\theta}_I w'_k - < \beta_p + \beta_d \cos\theta_I + \beta_s \sin\theta_I > +w'_k \cos\theta_I)\}]$$

$$+Im_{\eta\zeta}[\{\dot\phi_k\cos(\theta_G+\phi_k)\}\{\sin(\theta_G+\phi_k)(\dot\theta_z v'_k\cos\theta_I-\dot\theta_I v'_k<\beta_d\cos\theta_I+\beta_s\sin\theta_I>$$

$$-v'_k\dot\theta_I-\dot\theta_I v'_k w'_k-v'_k\cos\theta_I)+\cos(\theta_G+\phi_k)(-\dot\phi_k-\dot\theta_x\cos\psi_k-\dot\theta_y\sin\psi_k+\dot\theta_z w'_k\cos\theta_I$$

$$-\dot\theta_I\beta_d<\cos\theta_I+\sin\theta_I>-\dot\theta_I\beta_s<\sin\theta_I-\cos\theta_I>-\dot\theta_I+\dot\theta_I w'_k<\beta_d\cos\theta_I+\beta_s\sin\theta_I>$$

$$+\dot\theta_I w'_k-<\beta_p+\beta_d\cos\theta_I+\beta_s\sin\theta_I>+w'_k\cos\theta_I)\}]$$

$$-Im_{\eta\zeta}[\{\dot\phi_k\sin(\theta_G+\phi_k)\}\{\sin(\theta_G+\phi_k)(-\dot\phi_k$$

$$+\dot\theta_x\cos\psi_k+\dot\theta_y\sin\psi_k-\dot\theta_z w'_k\cos\theta_I+\dot\theta_I\beta_d\cos\theta_I$$

$$+\sin\theta_I>+\dot\theta_I\beta_s<\sin\theta_I-\cos\theta_I>+\dot\theta_I+\dot\theta_I w'_k<\beta_d\cos\theta_I+\beta_s\sin\theta_I>+\dot\theta_I w'_k$$

$$+<\beta_p+\beta_d\cos\theta_I+\beta_s\sin\theta_I>+w'_k\cos\theta_I)+\cos(\theta_G+\phi_k)(-\dot\theta_z v'_k\cos\theta_I$$

$$+v'_k\dot\theta_I<\beta_d\cos\theta_I+\beta_s\sin\theta_I>+\dot\theta_I v'_k+\dot\theta_I w'_k v'_k+v'_k\cos\theta_I)\}]$$

$$+m\eta_m[\dot\phi_k\cos(\theta_G+\phi_k)][\dot R_x\{-\sin\psi_k\cos\theta_I+\cos\psi_k(\beta_s\cos\theta_I-\beta_d\sin\theta_I)\}$$

$$+\dot R_y\{\cos\psi_k\cos\theta_I+\sin\psi_k(\beta_s\cos\theta_I-\beta_d\sin\theta_I)\}-\dot R_z\sin\theta_I$$

$$-\dot\theta_I\{(\beta_s\cos\theta_I+\beta_d\cos\theta_I)(e_1+e_2)\}+\dot v_k-a\dot\theta_I\sin\theta_I$$

$$-w_k(\dot\theta_x\cos\psi_k+\dot\theta_y\sin\psi_k)-\dot\theta_I w_k\{\beta_d(\cos\theta_I+\sin\theta_I)+\beta_s(\sin\theta_I-\cos\theta_I)+1\}$$

$$-\dot\theta_I\{(\beta_d\cos\theta_I+\beta_s\sin\theta_I)(2<e_1+e_2>+<n-1>l_e+x_k+u_k)$$

$$+a(\beta_s\cos\theta_I-\beta_d\sin\theta_I)$$

$$-(e_1+e_2)-(<n-1>l_e)-x_k-u_k\}-\{(\beta_p+\beta_d\cos\theta_I+\beta_s\sin\theta_I)w_k$$

$$-a\cos\theta_I(\beta_s\cos\theta_I-\beta_d\sin\theta_I)+\cos\theta_I(e_1+e_2+<n-1>l_e+x_k+u_k)\}]$$

$$-m\zeta_m[\dot R_x\{-\sin\psi_k\cos\theta_I+\cos\psi_k(\beta_s\cos\theta_I-\beta_d\sin\theta_I)\}$$

$$+\dot R_y\{\cos\psi_k\cos\theta_I+\sin\psi_k(\beta_s\cos\theta_I-\beta_d\sin\theta_I)\}-\dot R_z\sin\theta_I$$

$$-\dot\theta_I\{(\beta_s\cos\theta_I+\beta_d\cos\theta_I)(e_1+e_2)\}+\dot v_k-a\dot\theta_I\sin\theta_I$$

$$-w_k(\dot\theta_x\cos\psi_k+\dot\theta_y\sin\psi_k)-\dot\theta_I w_k\{\beta_d(\cos\theta_I+\sin\theta_I)+\beta_s(\sin\theta_I-\cos\theta_I)+1\}$$

$$-\dot\theta_I\{(\beta_d\cos\theta_I+\beta_s\sin\theta_I)(2<e_1+e_2>+<n-1>l_e+x_k+u_k)$$

$$+a(\beta_s\cos\theta_I-\beta_d\sin\theta_I)$$

$$-(e_1+e_2)-(<n-1>l_e)-x_k-u_k\}-\{(\beta_p+\beta_d\cos\theta_I+\beta_s\sin\theta_I)w_k$$

$$-a\cos\theta_I(\beta_s\cos\theta_I-\beta_d\sin\theta_I)+\cos\theta_I(e_1+e_2+<n-1>l_e+x_k+u_k)\}]$$
$$+Im_{\zeta\zeta}[\sin(\theta_G+\phi_k)][\sin(\theta_G+\phi_k)\{-\ddot{\phi}_k+(-\dot{\theta}_x+\ddot{\theta}_y)\sin\psi_k+(\ddot{\theta}_x+\dot{\theta}_y)\cos\psi_k$$
$$-\ddot{\theta}_z w_k'\cos\theta_I-\dot{\theta}_z\dot{w}_k'\cos\theta_I+\ddot{\theta}_I(\beta_d<\cos\theta_I+\sin\theta_I>+\beta_s<\sin\theta_I-\cos\theta_I>)$$
$$+\dot{\theta}_I^2(\beta_d<-\sin\theta_I+\cos\theta_I>+\beta_s<\sin\theta_I+\cos\theta_I>)+\ddot{\theta}_I$$
$$+(\ddot{\theta}_I w_k'+\dot{\theta}_I\dot{w}_k')(\dot{\theta}_I<-\beta_d\sin\theta_I+\beta_s\cos\theta_I>+1)+\dot{\theta}_I(-\beta_d\sin\theta_I+\beta_s\cos\theta_I)$$
$$+\dot{w}_k'\cos\theta_I-\dot{\phi}_k(-\dot{\theta}_z v_k'\cos\theta_I+v_k'q\dot{\theta}_I<\beta_d\cos\theta_I+\beta_s\sin\theta_I>$$
$$+\dot{\theta}_I v_k'+\dot{\theta}_I w_k'v_k'+v_k'\cos\theta_I)\}$$
$$+\cos(\theta_G+\phi_k)\{\dot{\phi}_k(-\dot{\phi}_k+\dot{\theta}_x\cos\psi_k+\dot{\theta}_y\sin\psi_k-\dot{\theta}_z w_k'\cos\theta_I+\dot{\theta}_I\beta_d<\cos\theta_I+\sin\theta_I>$$
$$+\dot{\theta}_I\beta_s<\sin\theta_I-\cos\theta_I>+\dot{\theta}_I+\dot{\theta}_I w_k'<\beta_d\cos\theta_I+\beta_s\sin\theta_I>$$
$$+\dot{\theta}_I w_k'+\beta_p+\beta_d\cos\theta_I+\beta_s\sin\theta_I+w_k'\cos\theta_I)-\ddot{\theta}_z v_k'\cos\theta_I-\dot{\theta}_z\dot{v}_k'\cos\theta_I$$
$$+(\ddot{\theta}_I v_k'+\dot{\theta}_I\dot{v}_k')(\dot{\theta}_I<-\beta_d\sin\theta_I+\beta_s\cos\theta_I>+1)$$
$$+\ddot{\theta}_I w_k'v_k'+\dot{\theta}_I(\dot{w}_k'v_k'+w_k'\dot{v}'_k)+\dot{v}'_k\cos\theta_I\}]$$
$$+Im_{\eta\eta}[\cos(\theta_G+\phi_k)][\sin(\theta_G+\phi_k)\{\ddot{\theta}_z v_k'\cos\theta_I+\dot{\theta}_z\dot{v}'_k-(\ddot{\theta}_I v_k'+\dot{\theta}_I\dot{v}'_k)$$
$$(\beta_d\cos\theta_I+\beta_s\sin\theta_I+1)-\ddot{\theta}_I v_k'w_k'-\dot{\theta}_I\dot{v}'_k w_k'-\dot{\theta}_I\dot{v}'_k\dot{w}_k'$$
$$-\dot{v}'_k\cos\theta_I+\dot{\phi}_k(\dot{\phi}_k+\dot{\theta}_x\cos\psi_k+\dot{\theta}_y\sin\psi_k-\dot{\theta}_z w_k'\cos\theta_I-\dot{\theta}_I\beta_d<\cos\theta_I+\sin\theta_I>$$
$$-\dot{\theta}_I\beta_s<\sin\theta_I-\cos\theta_I>-\dot{\theta}_I+\dot{\theta}_I w_k'<\beta_d\cos\theta_I+\beta_s\sin\theta_I>+\dot{\theta}_I w_k'$$
$$-<\beta_p+\beta_d\cos\theta_I+\beta_s\sin\theta_I>+w_k'\cos\theta_I)\}+\cos(\theta_G+\phi_k)\{\dot{\phi}_k(\dot{\theta}_z v_k'\cos\theta_I$$
$$-\dot{\theta}_I v_k'<\beta_d\cos\theta_I+\beta_s\sin\theta_I>-v_k'\dot{\theta}_I-\dot{\theta}_I v_k'w_k'-v_k'\cos\theta_I)-\ddot{\phi}_k$$
$$+(\dot{\theta}_x-\ddot{\theta}_y)\sin\psi_k-(\ddot{\theta}_x+\dot{\theta}_y)\cos\psi_k+\ddot{\theta}_z w_k'\cos\theta_I+\dot{\theta}_z\dot{w}_k'\cos\theta_I$$
$$-\dot{\theta}_I^2(\beta_d<-\sin\theta_I+\cos\theta_I>$$
$$+\beta_s<\cos\theta_I+\sin\theta_I>)-\ddot{\theta}_I(\beta_d<\cos\theta_I+\sin\theta_I>+\beta_s<\sin\theta_I-\cos\theta_I>$$
$$-\ddot{\theta}_I+\ddot{\theta}_I w_k'(\beta_d\cos\theta_I+\beta_s\sin\theta_I)+\dot{\theta}_I\dot{w}_k'(\beta_d\cos\theta_I+\beta_s\sin\theta_I)$$
$$+\dot{\theta}_I^2 w_k'(\beta_d\cos\theta_I+\beta_s\sin\theta_I)+\ddot{\theta}_I w_k'+\dot{\theta}_I\dot{w}_k'$$
$$-\dot{\theta}_I(-\beta_d\sin\theta_I+\beta_s\cos\theta_I)+\dot{w}_k'\cos\theta_I\}]$$

$$+Im_{\eta\zeta}[\sin(\theta_G + \phi_k)][\sin(\theta_G + \phi_k)\{\ddot{\theta}_z v'_k \cos\theta_I + \dot{\theta}_z \dot{v}'_k - (\ddot{\theta}_I v'_k + \dot{\theta}_I \dot{v}'_k)$$

$$(\beta_d \cos\theta_I + \beta_s \sin\theta_I + 1) - \ddot{\theta}_I v'_k w'_k - \dot{\theta}_I \dot{v}'_k w'_k - \dot{\theta}_I v'_k \dot{w}'_k$$

$$-\dot{v}'_k \cos\theta_I + \dot{\phi}_k(\dot{\phi}_k + \dot{\theta}_x \cos\psi_k + \dot{\theta}_y \sin\psi_k - \dot{\theta}_z w'_k \cos\theta_I - \dot{\theta}_I \beta_d < \cos\theta_I + \sin\theta_I >$$

$$-\dot{\theta}_I \beta_s < \sin\theta_I - \cos\theta_I > -\dot{\theta}_I + \dot{\theta}_I w'_k < \beta_d \cos\theta_I + \beta_s \sin\theta_I > +\dot{\theta}_I w'_k$$

$$- < \beta_p + \beta_d \cos\theta_I + \beta_s \sin\theta_I > +w'_k \cos\theta_I)\} + \cos(\theta_G + \phi_k)\{\dot{\phi}_k(\dot{\theta}_z v'_k \cos\theta_I$$

$$-\dot{\theta}_I v'_k < \beta_d \cos\theta_I + \beta_s \sin\theta_I > -v'_k \dot{\theta}_I - \dot{\theta}_I v'_k w'_k - v'_k \cos\theta_I) - \ddot{\phi}_k$$

$$+(\dot{\theta}_x - \ddot{\theta}_y) \sin\psi_k - (\ddot{\theta}_x + \dot{\theta}_y) \cos\psi_k + \ddot{\theta}_z w'_k \cos\theta_I + \dot{\theta}_z \dot{w}'_k \cos\theta_I$$

$$-\dot{\theta}_I^2(\beta_d < -\sin\theta_I + \cos\theta_I >$$

$$+\beta_s < \cos\theta_I + \sin\theta_I >) - \ddot{\theta}_I(\beta_d < \cos\theta_I + \sin\theta_I > +\beta_s < \sin\theta_I - \cos\theta_I >$$

$$-\ddot{\theta}_I + \ddot{\theta}_I w'_k(\beta_d \cos\theta_I + \beta_s \sin\theta_I) + \dot{\theta}_I \dot{w}'_k(\beta_d \cos\theta_I + \beta_s \sin\theta_I)$$

$$+\dot{\theta}_I^2 w'_k(\beta_d \cos\theta_I + \beta_s \sin\theta_I) + \ddot{\theta}_I w'_k + \dot{\theta}_I \dot{w}'_k$$

$$-\dot{\theta}_I(-\beta_d \sin\theta_I + \beta_s \cos\theta_I) + \dot{w}'_k \cos\theta_I\}]$$

$$+Im_{\eta\zeta}[\cos(\theta_G + \phi_k)][\sin(\theta_G + \phi_k)\{-\ddot{\phi}_k + (-\dot{\theta}_x + \ddot{\theta}_y) \sin\psi_k + (\ddot{\theta}_x + \dot{\theta}_y) \cos\psi_k$$

$$-\ddot{\theta}_z w'_k \cos\theta_I - \dot{\theta}_z \dot{w}'_k \cos\theta_I + \ddot{\theta}_I(\beta_d < \cos\theta_I + \sin\theta_I > +\beta_s < \sin\theta_I - \cos\theta_I >)$$

$$+\dot{\theta}_I^2(\beta_d < -\sin\theta_I + \cos\theta_I > +\beta_s < \sin\theta_I + \cos\theta_I >) + \ddot{\theta}_I$$

$$+(\ddot{\theta}_I w'_k + \dot{\theta}_I \dot{w}'_k)(\dot{\theta}_I < -\beta_d \sin\theta_I + \beta_s \cos\theta_I > +1) + \dot{\theta}_I(-\beta_d \sin\theta_I + \beta_s \cos\theta_I)$$

$$+\dot{w}'_k \cos\theta_I - \dot{\phi}_k(-\dot{\theta}_z v'_k \cos\theta_I + v'_k \dot{\theta}_I < \beta_d \cos\theta_I + \beta_s \sin\theta_I >$$

$$+\dot{\theta}_I v'_k + \dot{\theta}_I w'_k v'_k + v'_k \cos\theta_I)\}$$

$$+\cos(\theta_G + \phi_k)\{\dot{\phi}_k(-\dot{\phi}_k + \dot{\theta}_x \cos\psi_k + \dot{\theta}_y \sin\psi_k - \dot{\theta}_z w'_k \cos\theta_I + \dot{\theta}_I \beta_d < \cos\theta_I + \sin\theta_I >$$

$$+\dot{\theta}_I \beta_s < \sin\theta_I - \cos\theta_I > +\dot{\theta}_I + \dot{\theta}_I w'_k < \beta_d \cos\theta_I + \beta_s \sin\theta_I >$$

$$+\dot{\theta}_I w'_k + \beta_p + \beta_d \cos\theta_I + \beta_s \sin\theta_I + w'_k \cos\theta_I) - \ddot{\theta}_z v'_k \cos\theta_I - \dot{\theta}_z \dot{v}'_k \cos\theta_I$$

$$+(\ddot{\theta}_I v'_k + \dot{\theta}_I \dot{v}'_k)(\dot{\theta}_I < -\beta_d \sin\theta_I + \beta_s \cos\theta_I > +1)$$

$$+\ddot{\theta}_I w'_k v'_k + \dot{\theta}_I(\dot{w}'_k v'_k + w'_k \dot{v}'_k) + \dot{v}'_k \cos\theta_I\}]$$

$$+m\eta_m[\sin(\theta_G + \phi_k)][(\ddot{R}_x + \dot{R}_y)\{-\sin\psi_k \cos\theta_I + \cos\psi_k(\beta_s \cos\theta_I - \beta_d \sin\theta_I)\}$$

$$+(-\dot{R}_x + \ddot{R}_y)\{\cos\psi_k \cos\theta_I + \sin\psi_k(\beta_s \cos\theta_I - \beta_d \sin\theta_I)\} - \ddot{R}_z \sin\theta_I) - \dot{R}_z\dot{\theta}_I \cos\theta_I$$

$$-\dot{\theta}_I(\dot{R}_x \cos\psi_k + \dot{R}_y \sin\psi_k)(\beta_s \cos\theta_I + \beta_d \cos\theta_I)$$

$$-a\ddot{\theta}_I \sin\theta_I) - a\dot{\theta}_I \cos\theta_I - \ddot{\theta}_I(\beta_s \cos\theta_I + \beta_d \cos\theta_I)(e_1 + e_2) + \ddot{v}_k$$

$$-\dot{\theta}_I^2)\beta_s \cos\theta_I - \beta_d \sin\theta_I)(e_1 + e_2) - \dot{w}_k(\dot{\theta}_x \cos\psi_k + \dot{\theta}_y \sin\psi_k)$$

$$-w_k\{(-\dot{\theta}_x + \ddot{\theta}_y) \sin\psi_k + (\ddot{\theta}_x + \dot{\theta}_y) \cos\psi_k\}$$

$$-(\ddot{\theta}_I w_k + \dot{\theta}_I \dot{w}_k)\{\beta_d(\cos\theta_I + \sin\theta_I) + \beta_s(\sin\theta_I - \cos\theta_I) + 1\}$$

$$-\dot{\theta}_I^2 w_k\{\beta_d(-\sin\theta_I + \cos\theta_I) + \beta_s(\cos\theta_I + \sin\theta_I)\}$$

$$-\ddot{\theta}_I\{(\beta_d \cos\theta_I + \beta_s \sin\theta_I)(2 < e_1 + e_2 > + < n-1 > l_e + x_k + u_k)$$

$$+a(\beta_s \cos\theta_I - \beta_d \sin\theta_I) - (e_1 + e_2) - (< n-1 > l_e) - x_k - u_k\}$$

$$-\dot{\theta}_I\{\dot{\theta}_I(-\beta_d \sin\theta_I + \beta_s \cos\theta_I)(2 < e_1 + e_2 > + < n-1 > l_e + x_k + u_k\}$$

$$+(\beta_d \cos\theta_I + \beta_s \sin\theta_I)\dot{u}_k - a\dot{\theta}_I(\beta_s \sin\theta_I + \beta_d \cos\theta_I) - \dot{u}_k\}$$

$$-\{\dot{w}_k(\beta_p + \beta_d \cos\theta_I + \beta_s \sin\theta_I) + w_k\dot{\theta}_I(-\beta_d \sin\theta_I + \beta_s \cos\theta_I)$$

$$+a\dot{\theta}_I \cos\theta_I(\beta_s \sin\theta_I + \beta_d \cos\theta_I) + \dot{u}_k \cos\theta_I\}]$$

$$+m\zeta_m[(\ddot{R}_x + \dot{R}_y)\{-\sin\psi_k \cos\theta_I + \cos\psi_k(\beta_s \cos\theta_I - \beta_d \sin\theta_I)\}$$

$$+(-\dot{R}_x + \ddot{R}_y)\{\cos\psi_k \cos\theta_I + \sin\psi_k(\beta_s \cos\theta_I - \beta_d \sin\theta_I)\} - \ddot{R}_z \sin\theta_I) - \dot{R}_z\dot{\theta}_I \cos\theta_I$$

$$-\dot{\theta}_I(\dot{R}_x \cos\psi_k + \dot{R}_y \sin\psi_k)(\beta_s \cos\theta_I + \beta_d \cos\theta_I)$$

$$-a\ddot{\theta}_I \sin\theta_I) - a\dot{\theta}_I \cos\theta_I - \ddot{\theta}_I(\beta_s \cos\theta_I + \beta_d \cos\theta_I)(e_1 + e_2) + \ddot{v}_k$$

$$-\dot{\theta}_I^2)\beta_s \cos\theta_I - \beta_d \sin\theta_I)(e_1 + e_2) - \dot{w}_k(\dot{\theta}_x \cos\psi_k + \dot{\theta}_y \sin\psi_k)$$

$$-w_k\{(-\dot{\theta}_x + \ddot{\theta}_y) \sin\psi_k + (\ddot{\theta}_x + \dot{\theta}_y) \cos\psi_k\}$$

$$-(\ddot{\theta}_I w_k + \dot{\theta}_I \dot{w}_k)\{\beta_d(\cos\theta_I + \sin\theta_I) + \beta_s(\sin\theta_I - \cos\theta_I) + 1\}$$

$$-\dot{\theta}_I^2 w_k\{\beta_d(-\sin\theta_I + \cos\theta_I) + \beta_s(\cos\theta_I + \sin\theta_I)\}$$

$$-\ddot{\theta}_I\{(\beta_d \cos\theta_I + \beta_s \sin\theta_I)(2 < e_1 + e_2 > + < n-1 > l_e + x_k + u_k)$$

$$+a(\beta_s \cos\theta_I - \beta_d \sin\theta_I) - (e_1 + e_2) - (< n-1 > l_e) - x_k - u_k\}$$

$$-\dot{\theta}_I\{\dot{\theta}_I(-\beta_d \sin\theta_I + \beta_s \cos\theta_I)(2 < e_1 + e_2 > + < n-1 > l_e + x_k + u_k\}$$

$$+(\beta_d\cos\theta_I+\beta_s\sin\theta_I)\dot{u}_k-a\dot{\theta}_I(\beta_s\sin\theta_I+\beta_d\cos\theta_I)-\dot{u}_k\}$$

$$-\{\dot{w}_k(\beta_p+\beta_d\cos\theta_I+\beta_s\sin\theta_I)+w_k\dot{\theta}_I(-\beta_d\sin\theta_I+\beta_s\cos\theta_I)$$

$$+a\dot{\theta}_I\cos\theta_I(\beta_s\sin\theta_I+\beta_d\cos\theta_I)+\dot{u}_k\cos\theta_I\}]$$

$$+Im_{\zeta\zeta}[\dot{\phi}_k\sin(\theta_G+\phi_k)][\sin(\theta_G+\phi_k)\{-\dot{\theta}_x w'_k\sin\psi_k\cos\theta_I+\dot{\theta}_y w'_k\cos\psi_k\cos\theta_I$$

$$+\dot{\theta}_I w'_k(\beta_s\cos\theta_I-\beta_d\sin\theta_I+1)-w'_k\sin\theta_I\}$$

$$+\cos(\theta_G+\phi_k)\{+\dot{\phi}_k+\dot{\theta}_x(\cos\psi_k-v'_k\sin\psi_k\cos\theta_I)+\dot{\theta}_y(\sin\psi_k+v'_k\cos\psi_k\cos\theta_I)$$

$$+\dot{\theta}_I(\beta_d<\cos\theta_I+\sin\theta_I>+\beta_s<\sin\theta_I-\cos\theta_I>+1$$

$$+v'_k<\beta_s\cos\theta_I-\beta_d\sin\theta_I+1>)+\beta_p+\beta_d\cos\theta_I+\beta_s\sin\theta_I-v'_k\sin\theta_I\}]$$

$$+Im_{\eta\eta}[\dot{\phi}_k\cos(\theta_G+\phi_k)][\sin(\theta_G+\phi_k)\{-\dot{\phi}_k+\dot{\theta}_x(\cos\psi_k+v'_k\sin\psi_k\cos\theta_I)$$

$$+\dot{\theta}_y(\sin\psi_k-v'_k\cos\psi_k\cos\theta_I)+\dot{\theta}_I(\beta_d<\cos\theta_I+\sin\theta_I>+\beta_s<\sin\theta_I-\cos\theta_I>+1$$

$$-v'_k<\beta_s\cos\theta_I-\beta_d\sin\theta_I+1>)+\beta_p+\beta_d\cos\theta_I+\beta_s\sin\theta_I+v'_k\sin\theta_I\}$$

$$+\cos(\theta_G+\phi_k)\{-\dot{\theta}_x w'_k\sin\psi_k+\dot{\theta}_y w'_k\cos\psi_k+\dot{\theta}_I w'_k(\beta_s\cos\theta_I-\beta_d\sin\theta_I+1)-w'_k\sin\theta_I\}]$$

$$+Im_{\eta\zeta}[\dot{\phi}_k\sin(\theta_G+\phi_k)][\sin(\theta_G+\phi_k)\{-\dot{\phi}_k+\dot{\theta}_x(\cos\psi_k+v'_k\sin\psi_k\cos\theta_I)$$

$$+\dot{\theta}_y(\sin\psi_k-v'_k\cos\psi_k\cos\theta_I)+\dot{\theta}_I(\beta_d<\cos\theta_I+\sin\theta_I>+\beta_s<\sin\theta_I-\cos\theta_I>+1$$

$$-v'_k<\beta_s\cos\theta_I-\beta_d\sin\theta_I+1>)+\beta_p+\beta_d\cos\theta_I+\beta_s\sin\theta_I+v'_k\sin\theta_I\}$$

$$+\cos(\theta_G+\phi_k)\{-\dot{\theta}_x w'_k\sin\psi_k+\dot{\theta}_y w'_k\cos\psi_k+\dot{\theta}_I w'_k(\beta_s\cos\theta_I-\beta_d\sin\theta_I+1)-w'_k\sin\theta_I\}]$$

$$+Im_{\eta\zeta}[\dot{\phi}_k\cos(\theta_G+\phi_k)][\sin(\theta_G+\phi_k)\{-\dot{\theta}_x w'_k\sin\psi_k\cos\theta_I+\dot{\theta}_y w'_k\cos\psi_k\cos\theta_I$$

$$+\dot{\theta}_I w'_k(\beta_s\cos\theta_I-\beta_d\sin\theta_I+1)-w'_k\sin\theta_I\}$$

$$+\cos(\theta_G+\phi_k)\{+\dot{\phi}_k+\dot{\theta}_x(\cos\psi_k-v'_k\sin\psi_k\cos\theta_I)+\dot{\theta}_y(\sin\psi_k+v'_k\cos\psi_k\cos\theta_I)$$

$$+\dot{\theta}_I(\beta_d<\cos\theta_I+\sin\theta_I>+\beta_s<\sin\theta_I-\cos\theta_I>+1$$

$$+v'_k<\beta_s\cos\theta_I-\beta_d\sin\theta_I+1>)+\beta_p+\beta_d\cos\theta_I+\beta_s\sin\theta_I-v'_k\sin\theta_I\}]$$

$$+m\eta_m[\dot{\phi}_k\sin(\theta_G+\phi_k)][-\dot{R}_x\{\cos\psi_k(\beta_p+\beta_d\cos\theta_I+\beta_s\sin\theta_I)+\sin\psi_k\sin\theta_I\}$$

$$-\dot{R}_y\{\sin\psi_k(\beta_p+\beta_d\cos\theta_I+\beta_s\sin\theta_I)-\cos\psi_k\cos\theta_I\}+\dot{R}_z\cos\theta_I$$

$$+\dot{\theta}_I(\beta_d\sin\theta_I-\beta_s\cos\theta_I)(e_1+e_2)$$

$$-a\dot{\theta}_I \cos\theta_I + \dot{w}_k + \dot{\theta}_x\{\cos\psi_k(a\cos\theta_I + v_k)$$

$$+\sin\psi_k \cos\theta_I(e_1 + e_2 + < n-1 > l_e + x_k + u_k)\}$$

$$+\dot{\theta}_y\{\sin\psi_k(a\cos\theta_I + v_k) - \cos\psi_k\cos\theta_I(e_1 + e_2 + < n-1 > l_e + x_k + u_k)\}$$

$$+\dot{\theta}_I(\beta_d < \cos\theta_I + \sin\theta_I > +\beta_s < \sin\theta_I - \cos\theta_I > +1)(a + v_k)$$

$$+(e_1 + e_2)(\beta_s\cos\theta_I - \beta_d\sin\theta_I)$$

$$-(\beta_s\cos\theta_I - \beta_d\sin\theta_I + 1)(e_1 + e_2 + < n-1 > l_e + x_k + u_k)$$

$$+a(\beta_s\cos\theta_I - \beta_d\sin\theta_I))\} + a(\beta_p + \beta_d\cos\theta_I + \beta_s\sin\theta_I)$$

$$+\sin\theta_I)(e_1 + e_2 + < n-1 > l_e + x_k + u_k)]$$

$$+m\zeta_m[\dot{\phi}_k\cos\psi_k][-\dot{R}_x\{\cos\psi_k(\beta_p + \beta_d\cos\theta_I + \beta_s\sin\theta_I) + \sin\psi_k\sin\theta_I\}$$

$$-\dot{R}_y\{\sin\psi_k(\beta_p + \beta_d\cos\theta_I + \beta_s\sin\theta_I) - \cos\psi_k\cos\theta_I\}$$

$$+\dot{R}_z\cos\theta_I + \dot{\theta}_I(\beta_d\sin\theta_I - \beta_s\cos\theta_I)(e_1 + e_2) - a\dot{\theta}_I\cos\theta_I$$

$$+\dot{w}_k + \dot{\theta}_x\{\cos\psi_k(a\cos\theta_I + v_k) + \sin\psi_k\cos\theta_I(e_1 + e_2 + < n-1 > l_e + x_k + u_k)\}$$

$$+\dot{\theta}_y\{\sin\psi_k(a\cos\theta_I + v_k) - \cos\psi_k\cos\theta_I(e_1 + e_2 + < n-1 > l_e + x_k + u_k)\}$$

$$+\dot{\theta}_I(\beta_d < \cos\theta_I + \sin\theta_I > +\beta_s < \sin\theta_I - \cos\theta_I > +1)(a + v_k)$$

$$+(e_1 + e_2)(\beta_s\cos\theta_I - \beta_d\sin\theta_I) - (\beta_s\cos\theta_I - \beta_d\sin\theta_I + 1)$$

$$(e_1 + e_2 + < n-1 > l_e + x_k + u_k)$$

$$+a(\beta_s\cos\theta_I - \beta_d\sin\theta_I))\} + a(\beta_p + \beta_d\cos\theta_I + \beta_s\sin\theta_I)$$

$$+\sin\theta_I)(e_1 + e_2 + < n-1 > l_e + x_k + u_k)]$$

$$+Im_{\zeta\zeta}[-\cos(\theta_G + \phi_k)][\sin(\theta_G + \phi_k)\{\cos\theta_I(\sin\psi_k < -\ddot{\theta}_x - \dot{\theta}_y >$$

$$+\cos\psi_k < -\dot{\theta}_x + \ddot{\theta}_y >)w_k'$$

$$-\sin\theta_I\dot{w}_k + (-\dot{\theta}_x\sin\psi_k + \dot{\theta}_y\cos\psi_k)\dot{w}_k'\cos\theta_I + (\ddot{\theta}_I w_k' + \dot{\theta}_I\dot{w}_k')$$

$$(\beta_s\cos\theta_I - \beta_d\sin\theta_I + 1)$$

$$-\dot{\theta}_I^2 w_k'(\beta_s\sin\theta_I + \beta_d\cos\theta_I) - \dot{\phi}_k(\dot{\phi}_k + \dot{\theta}_x < \cos\psi_k - v_k'\sin\psi_k\cos\theta_I >$$

$$+\dot{\theta}_y < \sin\psi_k + v_k'\cos\psi_k\cos\theta_I >) - \dot{\phi}_k\dot{\theta}_I(\beta_d < \cos\theta_I + \sin\theta_I > +\beta_s < \sin\theta_I - \cos\theta_I >$$

$$+1+v_k' < \beta_s \cos\theta_I - \beta_d \sin\theta_I + 1 >) - \dot{\phi}_k v_k' \sin\theta_I) - \dot{\phi}_k(\beta_p + \beta_d \cos\theta_I + \beta_s \sin\theta_I)\}$$
$$+\cos(\theta_G + \phi_k)\{\dot{\phi}_k(-\dot{\theta}_x w_k' \sin\psi_k \cos\theta_I + \dot{\theta}_y w_k' \cos\psi_k \cos\theta_I$$
$$+\dot{\theta}_I w_k' < \beta_s \cos\theta_I - \beta_d \sin\theta_I + 1 >)$$
$$-\sin\theta_I w_k' + \ddot{\phi}_k + (\ddot{\theta}_x + \dot{\theta}_y)(\cos\psi_k - v_k' \sin\psi_k \cos\theta_I)$$
$$+(-\dot{\theta}_x + \ddot{\theta}_y)(\sin\psi_k + v_k' \cos\psi_k)$$
$$+(-\dot{\theta}_x \sin\psi_k + \dot{\theta}_y \cos\psi_k)\cos\theta_I \dot{v}'_k + \ddot{\theta}_I(\beta_d < \cos\theta_I + \sin\theta_I >$$
$$+\beta_s < \sin\theta_I - \cos\theta_I >$$
$$+1+v_k' < \beta_s \cos\theta_I - \beta_d \sin\theta_I + 1 >) + \dot{\theta}_I^2(\beta_d < -\sin\theta_I + \cos\theta_I >$$
$$+\beta_s < \cos\theta_I + \sin\theta_I > -v_k' < \beta_s \sin\theta_I + \beta_d \cos\theta_I >)$$
$$+\dot{\theta}_I \dot{v}'_k < \beta_s \cos\theta_I - \beta_d \sin\theta_I + 1 >$$
$$+\dot{\theta}_I(-\beta_d \sin\theta_I + \beta_s \cos\theta_I) + \dot{v}_k'\}]$$
$$+Im_{m}[\sin(\theta_G + \phi_k)][\sin(\theta_G + \phi_k)\{-\ddot{\phi}_k + (\ddot{\theta}_x + \dot{\theta}_y)(\cos\psi_k + v_k' \sin\psi_k \cos\theta_I)$$
$$+(-\dot{\theta}_x + \ddot{\theta}_y) + (\sin\psi_k - v_k' \cos\psi_k \cos\theta_I)$$
$$+(\dot{\theta}_x \sin\psi_k - \dot{\theta}_y \cos\psi_k)\cos\theta_I \dot{v}_k' + \ddot{\theta}_I(\beta_d < \cos\theta_I + \sin\theta_I >$$
$$+\beta_s < \sin\theta_I - \cos\theta_I > +1 - v_k' < \beta_s \cos\theta_I - \beta_d \sin\theta_I + 1 >)$$
$$+\dot{\theta}_I^2(\beta_d < -\sin\theta_I + \cos\theta_I > +\beta_s < \cos\theta_I + \sin\theta_I > +v_k' < \beta_s \sin\theta_I + \beta_d \cos\theta_I >)$$
$$-\dot{\theta}_I \dot{v}'_k(\beta_s \cos\theta_I - \beta_d \sin\theta_I + 1) + \dot{\theta}_I(-\beta_d \sin\theta_I + \beta_s \cos\theta_I) + \dot{v}_k' \sin\theta_I$$
$$-\dot{\phi}_k(-\dot{\theta}_x w_k' \sin\psi_k + \dot{\theta}_y w_k' \cos\psi_k$$
$$+\dot{\theta}_I w_k' < \beta_s \cos\theta_I - \beta_d \sin\theta_I + 1 >)\} + \cos(\theta_G + \phi_k)\{\dot{\phi}_k(-\dot{\phi}_k$$
$$+\dot{\theta}_x < \cos\psi_k + v_k' \sin\psi_k \cos\theta_I > +\dot{\theta}_y < \sin\psi_k - v_k' \cos\psi_k > \cos\theta_I)$$
$$+\dot{\theta}_I \dot{\phi}_k(\beta_d < \cos\theta_I + \sin\theta_I > +\beta_s < \sin\theta_I - \cos\theta_I >$$
$$+1 - v_k' < \beta_s \cos\theta_I - \beta_d \sin\theta_I + 1 >) + \dot{\phi}_k(\beta_p + \beta_d \cos\theta_I + \beta_s \sin\theta_I)$$
$$+(- < \ddot{\theta}_x + \dot{\theta}_y > \sin\psi_k + < -\dot{\theta}_x + \ddot{\theta}_y > \cos\psi_k) w_k'$$
$$+(-\dot{\theta}_x \sin\psi_k + \dot{\theta}_y \cos\psi_k)\dot{w}_k' + (\ddot{\theta}_I w_k' + \dot{\theta}_I \dot{w}_k')(\beta_s \cos\theta_I - \beta_d \sin\theta_I)$$

$$-\dot{w}_k' \sin\theta_I) - \dot{\theta}_I^2 w_k'(\beta_s \sin\theta_I + \beta_d \cos\theta_I)\}]$$
$$+Im_{\eta\zeta}[-\cos(\theta_G+\phi_k)][\sin(\theta_G+\phi_k)\{-\ddot{\phi}_k + (\ddot{\theta}_x + \dot{\theta}_y)(\cos\psi_k + v_k' \sin\psi_k \cos\theta_I)$$
$$+(-\dot{\theta}_x + \ddot{\theta}_y) + (\sin\psi_k - v_k' \cos\psi_k \cos\theta_I)$$
$$+(\dot{\theta}_x \sin\psi_k - \dot{\theta}_y \cos\psi_k)\cos\theta_I \dot{v}_k' + \ddot{\theta}_I(\beta_d < \cos\theta_I + \sin\theta_I >$$
$$+\beta_s < \sin\theta_I - \cos\theta_I > +1 - v_k' < \beta_s \cos\theta_I - \beta_d \sin\theta_I + 1 >)$$
$$+\dot{\theta}_I^2(\beta_d < -\sin\theta_I + \cos\theta_I > +\beta_s < \cos\theta_I + \sin\theta_I > +v_k' < \beta_s \sin\theta_I + \beta_d \cos\theta_I >)$$
$$-\dot{\theta}_I \dot{v'}_k(\beta_s \cos\theta_I - \beta_d \sin\theta_I + 1) + \dot{\theta}_I(-\beta_d \sin\theta_I + \beta_s \cos\theta_I) + \dot{v}_k' \sin\theta_I$$
$$-\dot{\phi}_k(-\dot{\theta}_x w_k' \sin\psi_k + \dot{\theta}_y w_k' \cos\psi_k$$
$$+\dot{\theta}_I w_k' < \beta_s \cos\theta_I - \beta_d \sin\theta_I + 1 >)\} + \cos(\theta_G+\phi_k)\{\dot{\phi}_k(-\dot{\phi}_k$$
$$+\dot{\theta}_x < \cos\psi_k + v_k' \sin\psi_k \cos\theta_I > +\dot{\theta}_y < \sin\psi_k - v_k' \cos\psi_k > \cos\theta_I)$$
$$+\dot{\theta}_I \dot{\phi}_k(\beta_d < \cos\theta_I + \sin\theta_I > +\beta_s < \sin\theta_I - \cos\theta_I >$$
$$+1 - v_k' < \beta_s \cos\theta_I - \beta_d \sin\theta_I + 1 >) + \dot{\phi}_k(\beta_p + \beta_d \cos\theta_I + \beta_s \sin\theta_I)$$
$$+(- < \ddot{\theta}_x + \dot{\theta}_y > \sin\psi_k + < -\dot{\theta}_x + \ddot{\theta}_y > \cos\psi_k) w_k'$$
$$+(-\dot{\theta}_x \sin\psi_k + \dot{\theta}_y \cos\psi_k)\dot{w}_k' + (\ddot{\theta}_I w_k' + \dot{\theta}_I \dot{w}_k')(\beta_s \cos\theta_I - \beta_d \sin\theta_I)$$
$$-\dot{w}_k' \sin\theta_I) - \dot{\theta}_I^2 w_k'(\beta_s \sin\theta_I + \beta_d \cos\theta_I)\}]$$
$$+Im_{\eta\zeta}[\sin(\theta_G+\phi_k)][\sin(\theta_G+\phi_k)\{\cos\theta_I(\sin\psi_k < -\ddot{\theta}_x - \dot{\theta}_y >$$
$$+\cos\psi_k < -\dot{\theta}_x + \ddot{\theta}_y >) w_k' - \sin\theta_I \dot{w}_k$$
$$+(-\dot{\theta}_x \sin\psi_k + \dot{\theta}_y \cos\psi_k)\dot{w}_k' \cos\theta_I + (\ddot{\theta}_I w_k' + \dot{\theta}_I \dot{w}_k')(\beta_s \cos\theta_I - \beta_d \sin\theta_I + 1)$$
$$-\dot{\theta}_I^2 w_k'(\beta_s \sin\theta_I + \beta_d \cos\theta_I) - \dot{\phi}_k(\dot{\phi}_k + \dot{\theta}_x < \cos\psi_k - v_k' \sin\psi_k \cos\theta_I >$$
$$+\dot{\theta}_y < \sin\psi_k + v_k' \cos\psi_k \cos\theta_I >) - \dot{\phi}_k \dot{\theta}_I(\beta_d < \cos\theta_I + \sin\theta_I > +\beta_s < \sin\theta_I - \cos\theta_I >$$
$$+1 + v_k' < \beta_s \cos\theta_I - \beta_d \sin\theta_I + 1 >) - \dot{\phi}_k v_k' \sin\theta_I) - \dot{\phi}_k(\beta_p + \beta_d \cos\theta_I + \beta_s \sin\theta_I)\}$$
$$+\cos(\theta_G+\phi_k)\{\dot{\phi}_k(-\dot{\theta}_x w_k' \sin\psi_k \cos\theta_I + \dot{\theta}_y w_k' \cos\psi_k \cos\theta_I$$
$$+\dot{\theta}_I w_k' < \beta_s \cos\theta_I - \beta_d \sin\theta_I + 1 >)$$
$$-\sin\theta_I w_k' + \ddot{\phi}_k + (\ddot{\theta}_x + \dot{\theta}_y)(\cos\psi_k - v_k' \sin\psi_k \cos\theta_I) + (-\dot{\theta}_x + \ddot{\theta}_y)(\sin\psi_k + v_k' \cos\psi_k)$$

$$+(-\dot{\theta}_x \sin\psi_k + \dot{\theta}_y \cos\psi_k)\cos\theta_I \dot{v}'_k + \ddot{\theta}_I(\beta_d < \cos\theta_I + \sin\theta_I > +\beta_s < \sin\theta_I - \cos\theta_I >$$

$$+1 + v'_k < \beta_s \cos\theta_I - \beta_d \sin\theta_I + 1 >) + \dot{\theta}_I^2(\beta_d < -\sin\theta_I + \cos\theta_I >$$

$$+\beta_s < \cos\theta_I + \sin\theta_I > -v'_k < \beta_s \sin\theta_I + \beta_d \cos\theta_I >)$$

$$+\dot{\theta}_I \dot{v}'_k < \beta_s \cos\theta_I - \beta_d \sin\theta_I + 1 >$$

$$+\dot{\theta}_I(-\beta_d \sin\theta_I + \beta_s \cos\theta_I) + \dot{v}'_k\}]$$

$$+m\eta_m[-\cos(\theta_G + \phi_k)][\{(\dot{R}_x - \ddot{R}_y)\sin\psi_k - (\ddot{R}_x + \dot{R}_y)\cos\psi_k\}$$

$$(\beta_p + \beta_d \cos\theta_I + \beta_s \sin\theta_I)$$

$$-\dot{\theta}_I(\dot{R}_x \cos\psi_k + \dot{R}_y \sin\psi_k)(-\beta_d \sin\theta_I + \beta_s \cos\theta_I) + \ddot{R}_z \cos\theta_I + \ddot{\theta}_I(\beta_d \sin\theta_I - \beta_s \cos\theta_I)$$

$$(e_1 + e_2) + \dot{\theta}_I^2(\beta_d \cos\theta_I + \beta_s \sin\theta_I) + \ddot{w}_k - a\ddot{\theta}_I \cos\theta_I + (\ddot{\theta}_x + \dot{\theta}_y)\{\cos\psi_k(a\cos\theta_I + v_k)$$

$$-\sin\psi_k \cos\theta_I(e_1 + e_2 + < n-1 > l_e + x_k + u_k)\} + (-\dot{\theta}_x + \ddot{\theta}_y)\{\sin\psi_k(a\cos\theta_I + v_k)$$

$$-\cos\psi_k \cos\theta_I(e_1 + e_2 + < n-1 > l_e + x_k + u_k)\} + \dot{\theta}_x(\dot{v}_k \cos\psi_k + \dot{u}_k \sin\psi_k)$$

$$+\dot{\theta}_y(\dot{v}_k \sin\psi_k - \dot{u}_k \cos\psi_k) + \ddot{\theta}_I\{(\beta_d < \cos\theta_I + \sin\theta_I > +\beta_s < \sin\theta_I - \cos\theta_I > +1)$$

$$(a\cos\theta_I + v_k) + (e_1 + e_2)(\beta_s \cos\theta_I - \beta_d \sin\theta_I) - (\beta_s \cos\theta_I - \beta_d \sin\theta_I + 1)$$

$$(e_1 + e_2 + < n-1 > l_e + x_k + u_k) + a(\beta_s \cos\theta_I - \beta_d \sin\theta_I)\}$$

$$+\dot{\theta}_I^2\{(\beta_d < -\sin\theta_I + \cos\theta_I > +\beta_s < \cos\theta_I + \sin\theta_I >)(a + v_k)$$

$$-(e_1 + e_2)(\beta_s \sin\theta_I + \beta_d \cos\theta_I) + (\beta_s \sin\theta_I + \beta_d \cos\theta_I)$$

$$(e_1 + e_2 + < n-1 > l_e + x_k + u_k)$$

$$-a(\beta_s \sin\theta_I + \beta_d \cos\theta_I)\} + \dot{\theta}_I\{(\beta_d < \cos\theta_I + \sin\theta_I >$$

$$+\beta_s < \sin\theta_I - \cos\theta_I > +1)\dot{v}_k - (\beta_s \cos\theta_I - \beta_d \sin\theta_I + 1)\dot{u}_k\}$$

$$+a\dot{\theta}_I(-\beta_d \sin\theta_I + \beta_s \cos\theta_I) + \dot{u}_k \sin\theta_I) + \dot{\theta}_I u_k \cos\theta_I]$$

$$+m\zeta_m[\sin(\theta_G + \phi_k)][\{(\dot{R}_x - \ddot{R}_y)\sin\psi_k - (\ddot{R}_x + \dot{R}_y)\cos\psi_k\}(\beta_p + \beta_d \cos\theta_I + \beta_s \sin\theta_I)$$

$$-\dot{\theta}_I(\dot{R}_x \cos\psi_k + \dot{R}_y \sin\psi_k)(-\beta_d \sin\theta_I + \beta_s \cos\theta_I) + \ddot{R}_z \cos\theta_I + \ddot{\theta}_I(\beta_d \sin\theta_I - \beta_s \cos\theta_I)$$

$$(e_1 + e_2) + \dot{\theta}_I^2(\beta_d \cos\theta_I + \beta_s \sin\theta_I) + \ddot{w}_k - a\ddot{\theta}_I \cos\theta_I + (\ddot{\theta}_x + \dot{\theta}_y)\{\cos\psi_k(a\cos\theta_I + v_k)$$

$$-\sin\psi_k \cos\theta_I(e_1 + e_2 + < n-1 > l_e + x_k + u_k)\} + (-\dot{\theta}_x + \ddot{\theta}_y)\{\sin\psi_k(a\cos\theta_I + v_k)$$

$$
\begin{aligned}
&-\cos\psi_k \cos\theta_I(e_1+e_2+<n-1>l_e+x_k+u_k)\}+\dot{\theta}_x(\dot{v}_k\cos\psi_k+\dot{u}_k\sin\psi_k)\\
&+\dot{\theta}_y(\dot{v}_k\sin\psi_k-\dot{u}_k\cos\psi_k)+\ddot{\theta}_I\{(\beta_d<\cos\theta_I+\sin\theta_I>+\beta_s<\sin\theta_I-\cos\theta_I>+1)\\
&(a\cos\theta_I+v_k)+(e_1+e_2)(\beta_s\cos\theta_I-\beta_d\sin\theta_I)-(\beta_s\cos\theta_I-\beta_d\sin\theta_I+1)\\
&(e_1+e_2+<n-1>l_e+x_k+u_k)+a(\beta_s\cos\theta_I-\beta_d\sin\theta_I)\}\\
&+\dot{\theta}_I\{(\beta_d<\cos\theta_I+\sin\theta_I>\\
&+\beta_s<\sin\theta_I-\cos\theta_I>+1)\dot{v}_k-(\beta_s\cos\theta_I-\beta_d\sin\theta_I+1)\dot{u}_k\}\\
&+a\dot{\theta}_I(-\beta_d\sin\theta_I+\beta_s\cos\theta_I)+\dot{u}_k\sin\theta_I)+\dot{\theta}_I u_k\cos\theta_I]
\end{aligned}
$$

$$\bar{Z}'_v =$$

$$Im_{\zeta\zeta}[\{\dot{\phi}_k \sin(\theta_G + \phi_k)\}\{\sin(\theta_G + \phi_k)(\dot{w}'_k + \dot{\theta}_x \sin\psi_k \cos\theta_I - \dot{\theta}_y \cos\psi_k \cos\theta_I$$

$$-\dot{\theta}_I < \beta_s \cos\theta_I - \beta_d \sin\theta_I - 1 > + \sin\theta_I)$$

$$+ \cos(\theta_G + \phi_k)(-\dot{v}'_k - \dot{\theta}_z - \dot{\theta}_I < 1 - \beta_d \cos\theta_I + \beta_s \sin\theta_I > - \cos\theta_I\}]$$

$$+Im_{\eta\eta}[\{\dot{\phi}_k \cos(\theta_G + \phi_k)\}\{\sin(\theta_G + \phi_k)(\dot{v}'_k - \dot{\theta}_z$$

$$-\dot{\theta}_I < 1 - \beta_d \cos\theta_I - \beta_s \sin\theta_I > - \cos\theta_I)$$

$$+ \cos(\theta_G + \phi_k)(-\dot{w}'_k + \dot{\theta}_x \sin\psi_k - \dot{\theta}_y \cos\psi_k + \dot{\theta}_I < \beta_s \cos\theta_I - \beta_d \sin\theta_I + 1 > - \sin\theta_I)\}]$$

$$+Im_{\eta\zeta}[\{\dot{\phi}_k \sin(\theta_G + \phi_k)\}\{\sin(\theta_G + \phi_k)(\dot{v}'_k - \dot{\theta}_z$$

$$-\dot{\theta}_I < 1 - \beta_d \cos\theta_I - \beta_s \sin\theta_I > - \cos\theta_I)$$

$$+ \cos(\theta_G + \phi_k)(-\dot{w}'_k + \dot{\theta}_x \sin\psi_k - \dot{\theta}_y \cos\psi_k + \dot{\theta}_I < \beta_s \cos\theta_I - \beta_d \sin\theta_I + 1 > - \sin\theta_I)\}]$$

$$+\{\dot{\phi}_k \cos(\theta_G + \phi_k)\}\{\sin(\theta_G + \phi_k)(\dot{w}'_k + \dot{\theta}_x \sin\psi_k \cos\theta_I - \dot{\theta}_y \cos\psi_k \cos\theta_I$$

$$-\dot{\theta}_I < \beta_s \cos\theta_I - \beta_d \sin\theta_I - 1 > + \sin\theta_I)$$

$$+ \cos(\theta_G + \phi_k)(-\dot{v}'_k - \dot{\theta}_z - \dot{\theta}_I < 1 - \beta_d \cos\theta_I + \beta_s \sin\theta_I > - \cos\theta_I\}]$$

$$+m\eta_m[\{\dot{\phi}_k \sin(\theta_G + \phi_k)\}\{\dot{R}_x \cos\psi_k + \dot{R}_y \sin\psi_k - \cos\theta_I(a\cos\theta_I + v_k)(1 + \dot{\theta}_I) - w_k \sin\theta_I\}]$$

$$+m\zeta_m[\{\dot{\phi}_k \cos(\theta_G + \phi_k)\}\{\dot{R}_x \cos\psi_k + \dot{R}_y \sin\psi_k - \cos\theta_I(a\cos\theta_I + v_k)(1 + \dot{\theta}_I) - w_k \sin\theta_I\}]$$

$$+Im_{\zeta\zeta}[\cos(\theta_G + \phi_k)\{-\dot{\theta}_z + \dot{\theta}_I(\beta_d \cos\theta_I + \beta_s \sin\theta_I + w'_k + 1) + \cos\theta_I\}]$$

$$[\sin(\theta_G + \phi_k)\{-\dot{\phi}_k + w'_k) + \dot{\theta}_I(\beta_d < \cos\theta_I + \sin\theta_I > +\beta_s < \sin\theta_I - \cos\theta_I > +1)$$

$$+(\beta_p + \beta_d \cos\theta_I + \beta_s \sin\theta_I)\} + v'_k \cos(\theta_G + \phi_k)(\dot{\theta}_I \cos\theta_I - 1)]$$

$$+Im_{\eta\eta}[\sin(\theta_G + \phi_k)\{\dot{\theta}_z - \dot{\theta}_I(\beta_d \cos\theta_I + \beta_s \sin\theta_I - w'_k + 1) - \cos\theta_I\}]$$

$$[\sin(\theta_G + \phi_k)\{v'_k(1 - \dot{\theta}_I \cos\theta_I)\} + \cos(\theta_G + \phi_k)\{(-\dot{\phi}_k + w'_k)$$

$$-\dot{\theta}_I(\beta_d < \cos\theta_I + \sin\theta_I > +\beta_s < \sin\theta_I - \cos\theta_I > +1) - (\beta_p + \beta_d \cos\theta_I + \beta_s \sin\theta_I)\}]$$

$$+Im_{\eta\zeta}[\cos(\theta_G + \phi_k)\{-\dot{\theta}_z + \dot{\theta}_I(\beta_d \cos\theta_I + \beta_s \sin\theta_I + w'_k + 1) + \cos\theta_I\}]$$

$$[\sin(\theta_G + \phi_k)\{v'_k(1 - \dot{\theta}_I)\} + \cos(\theta_G + \phi_k)\{(-\dot{\phi}_k + w'_k)$$

$$-\dot{\theta}_I(\beta_d < \cos\theta_I + \sin\theta_I > +\beta_s < \sin\theta_I - \cos\theta_I > +1) - (\beta_p + \beta_d\cos\theta_I + \beta_s\sin\theta_I)\}]$$

$$+Im_{\eta\zeta}[\sin(\theta_G + \phi_k)\{\dot{\theta}_z - \dot{\theta}_I((\beta_d\cos\theta_I + \beta_s\sin\theta_I - w'_k + 1) - \cos\theta_I\}]$$

$$[\sin(\theta_G + \phi_k)\{-\dot{\phi}_k + w'_k) + \dot{\theta}_I(\beta_d < \cos\theta_I + \sin\theta_I > +\beta_s < \sin\theta_I - \cos\theta_I > +1)$$

$$+(\beta_p + \beta_d\cos\theta_I + \beta_s\sin\theta_I)\} + v'_k\cos(\theta_G + \phi_k)(\dot{\theta}_I - 1)]$$

$$+m\eta_m[\cos(\theta_G + \phi_k)\{-\dot{\theta}_z + \dot{\theta}_I(\beta_d\cos\theta_I + \beta_s\sin\theta_I + w'_k + 1) + 1\}]$$

$$[-\dot{\theta}_I w_k - (e_1 + e_2 + < n-1 > l_e + x_k)(1 - \dot{\theta}_I\cos\theta_I)]$$

$$+m\zeta_m[\sin(\theta_G + \phi_k)\{\dot{\theta}_z - \dot{\theta}_I(\beta_d\cos\theta_I + \beta_s\sin\theta_I - w'_k + 1) - 1\}]$$

$$[-\dot{\theta}_I w_k - (e_1 + e_2 + < n-1 > l_e + x_k)(1 - \dot{\theta}_I\cos\theta_I)]$$

$$+Im_{\zeta\zeta}[\cos(\theta_G + \phi_k)\{\dot{\theta}_I(\beta_d\cos\theta_I + \beta_s\sin\theta_I + w'_k + 1) + \cos\theta_I\}$$

$$\{\sin(\theta_G + \phi_k)(\dot{\theta}_x\cos\psi_k + \dot{\theta}_y\sin\psi_k)\}]$$

$$+Im_{\eta\eta}[\sin(\theta_G + \phi_k)\{-\dot{\theta}_I(\beta_d\cos\theta_I + \beta_s\sin\theta_I - w'_k + 1) - \cos\theta_I\}$$

$$\{-\cos(\theta_G + \phi_k)(\dot{\theta}_x\cos\psi_k + \dot{\theta}_y\sin\psi_k)\}]$$

$$+Im_{\eta\zeta}[\cos(\theta_G + \phi_k)\{\dot{\theta}_I(\beta_d\cos\theta_I + \beta_s\sin\theta_I + w'_k + 1) + \cos\theta_I\}$$

$$\{-\cos(\theta_G + \phi_k)(\dot{\theta}_x\cos\psi_k + \dot{\theta}_y\sin\psi_k)\}$$

$$+\sin(\theta_G + \phi_k)\{-\dot{\theta}_I(\beta_d\cos\theta_I + \beta_s\sin\theta_I - w'_k + 1) - \cos\theta_I\}$$

$$\{\sin(\theta_G + \phi_k)(\dot{\theta}_x\cos\psi_k + \dot{\theta}_y\sin\psi_k)\}]$$

$$+m\eta_m[\cos(\theta_G + \phi_k)\{\dot{\theta}_I(\beta_d\cos\theta_I + \beta_s\sin\theta_I + w'_k + 1) + 1\}$$

$$\{-\dot{R}_x\sin\psi_k\cos\theta_I + \dot{R}_y\cos\psi_k\cos\theta_I$$

$$+(\dot{\theta}_z\cos\theta_I + \beta_d\cos\theta_I + \beta_s\sin\theta_I)(< n-1 > l_e + x_k)\}]$$

$$+m\zeta_m[\sin(\theta_G + \phi_k)\{-\dot{\theta}_I(\beta_d\cos\theta_I + \beta_s\sin\theta_I - w'_k + 1) - 1\}$$

$$\{-\dot{R}_x\sin\psi_k\cos\theta_I + \dot{R}_y\cos\psi_k\cos\theta_I$$

$$+(\dot{\theta}_z\cos\theta_I + \beta_d\cos\theta_I + \beta_s\sin\theta_I)(< n-1 > l_e + x_k)\}]$$

$$+Im_{\zeta\zeta}[\cos(\theta_G + \phi_k)(1 + \dot{\theta}_I)][\sin(\theta_G + \phi_k)\{-\dot{\theta}_z w'_k\cos\theta_I + \dot{\theta}_I w'_k(\beta_d\cos\theta_I + \beta_s\sin\theta_I)\}$$

$$+\cos(\theta_G + \phi_k)\{-\dot{\theta}_z v'_k\cos\theta_I + \dot{\theta}_I v'_k(\beta_d\cos\theta_I + \beta_s\sin\theta_I) - \dot{\theta}_I w'_k v'_k\}]$$

$$+Im_{\eta\eta}[-\sin(\theta_G+\phi_k)(1+\dot{\theta}_I)][\sin(\theta_G+\phi_k)\{\dot{\theta}_z v_k'\cos\theta_I-\dot{\theta}_I v_k'(\beta_d\cos\theta_I+\beta_s\sin\theta_I)$$

$$+\dot{\theta}_I w_k' v_k'\}+\cos(\theta_G+\phi_k)\{\dot{\theta}_z w_k'\cos\theta_I+\dot{\theta}_I w_k'(\beta_d\cos\theta_I+\beta_s\sin\theta_I)\}]$$

$$+Im_{\eta\zeta}[\cos(\theta_G+\phi_k)(1+\dot{\theta}_I)][\sin(\theta_G+\phi_k)\{\dot{\theta}_z v_k'\cos\theta_I-\dot{\theta}_I v_k'(\beta_d\cos\theta_I+\beta_s\sin\theta_I)$$

$$+\dot{\theta}_I w_k' v_k'\}+\cos(\theta_G+\phi_k)\{\dot{\theta}_z w_k'\cos\theta_I+\dot{\theta}_I w_k'(\beta_d\cos\theta_I+\beta_s\sin\theta_I)\}]$$

$$+Im_{\eta\zeta}[-\sin(\theta_G+\phi_k)(1+\dot{\theta}_I)][\sin(\theta_G+\phi_k)\{-\dot{\theta}_z w_k'\cos\theta_I+\dot{\theta}_I w_k'(\beta_d\cos\theta_I+\beta_s\sin\theta_I)\}$$

$$+\cos(\theta_G+\phi_k)\{-\dot{\theta}_z v_k'\cos\theta_I+\dot{\theta}_I v_k'(\beta_d\cos\theta_I+\beta_s\sin\theta_I)-\dot{\theta}_I w_k' v_k'\}]$$

$$+m\eta_m[\cos(\theta_G+\phi_k)(1+\dot{\theta}_I)][(\dot{R}_x\cos\psi_k+\dot{R}_y\sin\psi_k)(\beta_s\cos\theta_I-\beta_d\sin\theta_I)$$

$$-\dot{\theta}_I(\beta_s\sin\theta_I+\beta_d\cos\theta_I)(e_1+e_2+2a\sin\theta_I)-w_k(\dot{\theta}_x\cos\psi_k+\dot{\theta}_y\sin\psi_k)$$

$$+\dot{\theta}_z\cos\theta_I(e_1+e_2)$$

$$+\dot{\theta}_I(a\cos\theta_I<\beta_s\cos\theta_I-\beta_d\sin\theta_I>-u_k)-(\beta_p+\beta_d\cos\theta_I+\beta_s\sin\theta_I)w_k$$

$$+a\cos\theta_I(\beta_s\cos\theta_I-\beta_d\sin\theta_I)]$$

$$+m\zeta_m[-\sin(\theta_G+\phi_k)(1+\dot{\theta}_I)][(\dot{R}_x\cos\psi_k+\dot{R}_y\sin\psi_k)(\beta_s\cos\theta_I-\beta_d\sin\theta_I)$$

$$-\dot{\theta}_I(\beta_s\sin\theta_I+\beta_d\cos\theta_I)(e_1+e_2+2a\sin\theta_I)-w_k(\dot{\theta}_x\cos\psi_k+\dot{\theta}_y\sin\psi_k)$$

$$+\dot{\theta}_z\cos\theta_I(e_1+e_2)$$

$$+\dot{\theta}_I(a\cos\theta_I<\beta_s\cos\theta_I-\beta_d\sin\theta_I>-u_k)-(\beta_p+\beta_d\cos\theta_I+\beta_s\sin\theta_I)w_k$$

$$+a\cos\theta_I(\beta_s\cos\theta_I-\beta_d\sin\theta_I)]$$

$$+Im_{\zeta\zeta}[\cos(\theta_G+\phi_k)(\cos\theta_I+\sin\theta_I)\{-\dot{\theta}_x\sin\psi_k+\dot{\theta}_y\cos\psi_k+\dot{\theta}_I(\beta_s\cos\theta_I-\beta_d\sin\theta_I)\}]$$

$$[\dot{\theta}_I w_k'\sin(\theta_G+\phi_k)+\cos(\theta_G+\phi_k)\{-\dot{\phi}_k+\dot{\theta}_I(\beta_d<\cos\theta_I+\sin\theta_I>$$

$$+\beta_s<\sin\theta_I-\cos\theta_I>+1)+\beta_p+\beta_d\cos\theta_I+\beta_s\sin\theta_I+\dot{\theta}_I v_k'\}]$$

$$+Im_{\eta\eta}[\sin(\theta_G+\phi_k)(\cos\theta_I+\sin\theta_I)\{\dot{\theta}_x\sin\psi_k-\dot{\theta}_y\cos\psi_k-\dot{\theta}_I(\beta_s\cos\theta_I-\beta_d\sin\theta_I)\}]$$

$$[\sin(\theta_G+\phi_k)\{-\dot{\phi}_k+\dot{\theta}_I(\beta_d<\cos\theta_I+\sin\theta_I>+\beta_s<\sin\theta_I-\cos\theta_I>+1)$$

$$+\beta_p+\beta_d\cos\theta_I+\beta_s\sin\theta_I-\dot{\theta}_I v_k'\}+\dot{\theta}_I w_k'\cos(\theta_G+\phi_k)]$$

$$+Im_{\eta\zeta}[\cos(\theta_G+\phi_k)(\cos\theta_I+\sin\theta_I)\{-\dot{\theta}_x\sin\psi_k+\dot{\theta}_y\cos\psi_k+\dot{\theta}_I(\beta_s\cos\theta_I-\beta_d\sin\theta_I)\}]$$

$$[\sin(\theta_G+\phi_k)\{-\dot{\phi}_k+\dot{\theta}_I(\beta_d<\cos\theta_I+\sin\theta_I>+\beta_s<\sin\theta_I-\cos\theta_I>+1)$$

$$+\beta_p + \beta_d \cos\theta_I + \beta_s \sin\theta_I - \dot{\theta}_I v'_k\} + \dot{\theta}_I w'_k \cos(\theta_G + \phi_k)]$$

$$+Im_{\eta\zeta}[\sin(\theta_G + \phi_k)(\cos\theta_I + \sin\theta_I)\{\dot{\theta}_x \sin\psi_k - \dot{\theta}_y \cos\psi_k - \dot{\theta}_I(\beta_s \cos\theta_I - \beta_d \sin\theta_I)\}]$$

$$[\dot{\theta}_I w'_k \sin(\theta_G + \phi_k) + \cos(\theta_G + \phi_k)\{-\dot{\phi}_k + \dot{\theta}_I(\beta_d < \cos\theta_I + \sin\theta_I >$$

$$+\beta_s < \sin\theta_I - \cos\theta_I > +1) + \beta_p + \beta_d \cos\theta_I + \beta_s \sin\theta_I + \dot{\theta}_I v'_k\}]$$

$$+m\eta_m[\cos(\theta_G + \phi_k)(\cos\theta_I + \sin\theta_I)\{-\dot{\theta}_x \sin\psi_k + \dot{\theta}_y \cos\psi_k + \dot{\theta}_I(\beta_s \cos\theta_I - \beta_d \sin\theta_I)\}$$

$$\{\ddot{w}_k + a\cos\theta_I + v_k - (\beta_s \cos\theta_I - \beta_d \sin\theta_I + 1)(< n-1 > l_e + x_k) - (e_1 + e_2)\}]$$

$$+m\zeta_m[\sin(\theta_G + \phi_k)(\cos\theta_I + \sin\theta_I)\{\dot{\theta}_x \sin\psi_k - \dot{\theta}_y \cos\psi_k - \dot{\theta}_I(\beta_s \cos\theta_I - \beta_d \sin\theta_I)\}$$

$$\{\ddot{w}_k + a\cos\theta_I + v_k - (\beta_s \cos\theta_I - \beta_d \sin\theta_I + 1)(< n-1 > l_e + x_k) - (e_1 + e_2)\}]$$

$$+Im_{\zeta\zeta}[\{\cos(\theta_G + \phi_k)(\cos\theta_I + \sin\theta_I)(\dot{\theta}_I < \beta_s \cos\theta_I - \beta_d \sin\theta_I >)\}$$

$$\{\cos(\theta_G + \phi_k)\cos\theta_I(\dot{\theta}_x \cos\psi_k + \dot{\theta}_y \sin\psi_k)\}]$$

$$+Im_{\eta\eta}[\{\sin(\theta_G + \phi_k)(\cos\theta_I + \sin\theta_I)(-\dot{\theta}_I < \beta_s \cos\theta_I - \beta_d \sin\theta_I >)\}$$

$$\{\sin(\theta_G + \phi_k)\cos\theta_I(\dot{\theta}_x \cos\psi_k + \dot{\theta}_y \sin\psi_k)\}]$$

$$+Im_{\eta\zeta}[\{\cos(\theta_G + \phi_k)(\cos\theta_I + \sin\theta_I)(\dot{\theta}_I < \beta_s \cos\theta_I - \beta_d \sin\theta_I >)\}$$

$$\{\sin(\theta_G + \phi_k)\cos\theta_I(\dot{\theta}_x \cos\psi_k + \dot{\theta}_y \sin\psi_k)\}$$

$$+\{\sin(\theta_G + \phi_k)(\cos\theta_I + \sin\theta_I)(-\dot{\theta}_I < \beta_s \cos\theta_I - \beta_d \sin\theta_I >)\}$$

$$\{\cos(\theta_G + \phi_k)\cos\theta_I(\dot{\theta}_x \cos\psi_k + \dot{\theta}_y \sin\psi_k)\}]$$

$$+m\eta_m[\{\cos(\theta_G + \phi_k)(\cos\theta_I + \sin\theta_I)(\dot{\theta}_I < \beta_s \cos\theta_I - \beta_d \sin\theta_I >)\}$$

$$\{\dot{R}_z \cos\theta_I + \dot{\theta}_x \sin\psi_k - \dot{\theta}_y \cos\psi_k)(< n-1 > l_e + x_k)\}]$$

$$+m\zeta_m[\{\sin(\theta_G + \phi_k)(\cos\theta_I + \sin\theta_I)(-\dot{\theta}_I < \beta_s \cos\theta_I - \beta_d \sin\theta_I >)\}$$

$$\{\dot{R}_z \cos\theta_I + \dot{\theta}_x \sin\psi_k - \dot{\theta}_y \cos\psi_k)(< n-1 > l_e + x_k)\}]$$

$$-Im_{\zeta\zeta}[\dot{\phi}_k \sin(\theta_G + \phi_k)\{\sin(\theta_G + \phi_k)(v'_k\dot{\phi}_k - \dot{w}'_k + \dot{\theta}_x \sin\psi_k \cos\theta_I - \dot{\theta}_y \cos\psi_k \cos\theta_I$$

$$-\dot{\theta}_z \sin\theta_I) - \dot{\theta}_I < \beta_s \cos\theta_I - \beta_d \sin\theta_I > +\dot{\theta}_I + \sin\theta_I)$$

$$+\cos(\theta_G + \phi_k)(-\dot{v}'_k - w'_k\dot{\phi}_k - \dot{\theta}_z \cos\theta_I - \dot{\theta}_I < 1 - \beta_d \cos\theta_I - \beta_s \sin\theta_I > -\cos\theta_I)\}]$$

$$-Im_{\eta\eta}[\dot{\phi}_k \cos(\theta_G + \phi_k)\{\sin(\theta_G + \phi_k)(\dot{v}'_k + w'_k\dot{\phi}_k - \dot{\theta}_z \cos\theta_I - \dot{\theta}_I < 1 - \beta_d \cos\theta_I$$

$$-\beta_s \sin\theta_I > -\cos\theta_I) + \cos(\theta_G + \phi_k)(v'_k\dot{\phi}_k - \dot{w}'_k - \dot{\theta}_x \sin\psi_k \cos\theta_I + \dot{\theta}_y \cos\psi_k \cos\theta_I$$

$$+\dot{\theta}_z \sin\theta_I) - \dot{\theta}_I < -\beta_s \cos\theta_I + \beta_d \sin\theta_I + \dot{\theta}_I > -\sin\theta_I))\}]$$

$$-Im_{\eta\zeta}[\dot{\phi}_k \sin(\theta_G + \phi_k)\{\sin(\theta_G + \phi_k)(\dot{v}'_k + w'_k\dot{\phi}_k - \dot{\theta}_z \cos\theta_I - \dot{\theta}_I < 1 - \beta_d \cos\theta_I$$

$$-\beta_s \sin\theta_I > -\cos\theta_I) + \cos(\theta_G + \phi_k)(v'_k\dot{\phi}_k - \dot{w}'_k - \dot{\theta}_x \sin\psi_k \cos\theta_I + \dot{\theta}_y \cos\psi_k \cos\theta_I$$

$$+\dot{\theta}_z \sin\theta_I) - \dot{\theta}_I < -\beta_s \cos\theta_I + \beta_d \sin\theta_I + \dot{\theta}_I > -\sin\theta_I))\}$$

$$+\dot{\phi}_k \cos(\theta_G + \phi_k)\{\sin(\theta_G + \phi_k)(v'_k\dot{\phi}_k - \dot{w}'_k + \dot{\theta}_x \sin\psi_k \cos\theta_I - \dot{\theta}_y \cos\psi_k \cos\theta_I$$

$$-\dot{\theta}_z \sin\theta_I) - \dot{\theta}_I < \beta_s \cos\theta_I - \beta_d \sin\theta_I > +\dot{\theta}_I + \sin\theta_I)$$

$$+\cos(\theta_G + \phi_k)(-\dot{v}'_k - w'_k\dot{\phi}_k - \dot{\theta}_z \cos\theta_I - \dot{\theta}_I < 1 - \beta_d \cos\theta_I - \beta_s \sin\theta_I > -\cos\theta_I))\}]$$

$$-m\eta_m[\dot{\phi}_k \sin(\theta_G + \phi_k)\{\dot{R}_x(\cos\psi_k - \sin\psi_k < -\beta_s \cos\theta_I + \beta_d \sin\theta_I >)$$

$$+\dot{R}_y(\sin\psi_k + \cos\psi_k < -\beta_s \cos\theta_I + \beta_d \sin\theta_I >)$$

$$+\dot{R}_z(\beta_p + \beta_d \cos\theta_I + \beta_s \sin\theta_I) + a\dot{\theta}_I(\beta_s \sin\theta_I + \beta_d \cos\theta_I) + \dot{u}_k - \dot{\theta}_x w_k \sin\psi_k \cos\theta_I$$

$$+\dot{\theta}_y w_k \cos\psi_k \cos\theta_I - \dot{\theta}_z \cos\theta_I(a\cos\theta_I + v_k)$$

$$-\dot{\theta}_I(< -w_k + e_1 + e_2 >< \beta_s \cos\theta_I - \beta_d \sin\theta_I >$$

$$+ < v_k + a\cos\theta_I) >< 1 - \beta_d \cos\theta_I - \beta_s \sin\theta_I > + < e_1 + e_2 >< \beta_d \cos\theta_I + \beta_s \sin\theta_I >$$

$$+w_k) - \cos\theta_I(a\cos\theta_I + < e_1 + e_2 >< \beta_s \cos\theta_I - \beta_d \sin\theta_I > +v_k) - \sin\theta_I)$$

$$(< e_1 + e_2 >< \beta_d \cos\theta_I + \beta_s \sin\theta_I > +w_k)\}]$$

$$-m\zeta_m[\dot{\phi}_k \cos(\theta_G + \phi_k)\{\dot{R}_x(\cos\psi_k - \sin\psi_k < -\beta_s \cos\theta_I + \beta_d \sin\theta_I >)$$

$$+\dot{R}_y(\sin\psi_k + \cos\psi_k < -\beta_s \cos\theta_I + \beta_d \sin\theta_I >)$$

$$+\dot{R}_z(\beta_p + \beta_d \cos\theta_I + \beta_s \sin\theta_I) + a\dot{\theta}_I(\beta_s \sin\theta_I + \beta_d \cos\theta_I) + \dot{u}_k - \dot{\theta}_x w_k \sin\psi_k \cos\theta_I$$

$$+\dot{\theta}_y w_k \cos\psi_k \cos\theta_I - \dot{\theta}_z \cos\theta_I(a\cos\theta_I + v_k)$$

$$-\dot{\theta}_I(< -w_k + e_1 + e_2 >< \beta_s \cos\theta_I - \beta_d \sin\theta_I >$$

$$+ < v_k + a\cos\theta_I) >< 1 - \beta_d \cos\theta_I - \beta_s \sin\theta_I > + < e_1 + e_2 >< \beta_d \cos\theta_I + \beta_s \sin\theta_I >$$

$$+w_k) - \cos\theta_I(a\cos\theta_I + < e_1 + e_2 >< \beta_s \cos\theta_I - \beta_d \sin\theta_I > +v_k) - \sin\theta_I)$$

$$(< e_1 + e_2 >< \beta_d \cos\theta_I + \beta_s \sin\theta_I > +w_k)\}]$$

$$-Im_{\zeta\zeta}[-\cos(\theta_G+\phi_k)][\sin(\theta_G+\phi_k)\{v_k'\ddot{\phi}_k+2\dot{v}'_k\dot{\phi}_k-\ddot{w}_k'+w_k'\dot{\phi}_k^2$$
$$-\cos\theta_I(<\ddot{\theta}_x+\dot{\theta}_y>\sin\psi_k+<\dot{\theta}_x-\ddot{\theta}_y>\cos\psi_k)-\dot{\phi}_k\dot{\theta}_z\cos\theta_I$$
$$+\ddot{\theta}_I(<\beta_s\cos\theta_I-\beta_d\sin\theta_I>-1)+\dot{\theta}_I(-\dot{\theta}_I<\beta_s\sin\theta_I-\beta_d\cos\theta_I>$$
$$+\dot{\phi}_k<-1+\beta_d\cos\theta_I+\beta_s\sin\theta_I>)-\ddot{\phi}_k\cos\theta_I\}+\cos(\theta_G+\phi_k)\{-\ddot{v}_k'+v_k\dot{\phi}_k^2$$
$$-2\dot{w}_k'\dot{\phi}_k-w_k'\ddot{\phi}_k-\dot{\phi}_k\cos\theta_I<\dot{\theta}_x\sin\psi_k-\dot{\theta}_y\cos\psi_k>+\ddot{\theta}_z\cos\theta_I$$
$$+\ddot{\theta}_I(1-<\beta_d\cos\theta_I+\beta_s\sin\theta_I>)$$
$$+\dot{\theta}_I(\dot{\phi}_k<\beta_s\cos\theta_I-\beta_d\sin\theta_I>+\dot{\theta}_I<\beta_d\sin\theta_I-\beta_s\cos\theta_I>-\dot{\phi}_k)+\ddot{\phi}_k\sin\theta_I\}]$$
$$-Im_{\eta\eta}[\sin(\theta_G+\phi_k)][\sin(\theta_G+\phi_k)\{\ddot{v}_k'-v_k\dot{\phi}_k^2+2\dot{w}_k'\dot{\phi}_k+w_k'\ddot{\phi}_k$$
$$-\dot{\phi}_k\cos\theta_I<\dot{\theta}_x\sin\psi_k-\dot{\theta}_y\cos\psi_k>+\ddot{\theta}_z\cos\theta_I+\ddot{\theta}_I(1-<\beta_d\cos\theta_I+\beta_s\sin\theta_I>)$$
$$+\dot{\theta}_I(\dot{\phi}_k<\beta_s\cos\theta_I-\beta_d\sin\theta_I>+\dot{\theta}_I<\beta_d\sin\theta_I-\beta_s\cos\theta_I>-\dot{\phi}_k)+\ddot{\phi}_k\sin\theta_I\}$$
$$+\cos(\theta_G+\phi_k)\{v_k'\ddot{\phi}_k+2\dot{v}'_k\dot{\phi}_k-\ddot{w}_k'+w_k'\dot{\phi}_k^2$$
$$+\cos\theta_I(<\ddot{\theta}_x+\dot{\theta}_y>\sin\psi_k+<\dot{\theta}_x-\ddot{\theta}_y>\cos\psi_k)+\dot{\phi}_k\dot{\theta}_z\cos\theta_I$$
$$+\ddot{\theta}_I(-\beta_s\cos\theta_I+\beta_d\sin\theta_I+1)+\dot{\theta}_I(\dot{\theta}_I<\beta_s\sin\theta_I-\beta_d\cos\theta_I>$$
$$+\dot{\phi}_k<1-\beta_d\cos\theta_I+\beta_s\sin\theta_I>)+\ddot{\phi}_k\cos\theta_I\}]$$
$$-Im_{\eta\zeta}[-\cos(\theta_G+\phi_k)][\sin(\theta_G+\phi_k)\{\ddot{v}_k'-v_k\dot{\phi}_k^2+2\dot{w}_k'\dot{\phi}_k+w_k'\ddot{\phi}_k$$
$$-\dot{\phi}_k\cos\theta_I<\dot{\theta}_x\sin\psi_k-\dot{\theta}_y\cos\psi_k>+\ddot{\theta}_z\cos\theta_I+\ddot{\theta}_I(1-<\beta_d\cos\theta_I+\beta_s\sin\theta_I>)$$
$$+\dot{\theta}_I(\dot{\phi}_k<\beta_s\cos\theta_I-\beta_d\sin\theta_I>+\dot{\theta}_I<\beta_d\sin\theta_I-\beta_s\cos\theta_I>-\dot{\phi}_k)+\ddot{\phi}_k\sin\theta_I\}$$
$$+\cos(\theta_G+\phi_k)\{v_k'\ddot{\phi}_k+2\dot{v}'_k\dot{\phi}_k-\ddot{w}_k'+w_k'\dot{\phi}_k^2$$
$$+\cos\theta_I(<\ddot{\theta}_x+\dot{\theta}_y>\sin\psi_k+<\dot{\theta}_x-\ddot{\theta}_y>\cos\psi_k)+\dot{\phi}_k\dot{\theta}_z\cos\theta_I$$
$$+\ddot{\theta}_I(-\beta_s\cos\theta_I+\beta_d\sin\theta_I+1)+\dot{\theta}_I(\dot{\theta}_I<\beta_s\sin\theta_I-\beta_d\cos\theta_I>$$
$$+\dot{\phi}_k<1-\beta_d\cos\theta_I+\beta_s\sin\theta_I>)+\ddot{\phi}_k\cos\theta_I\}]$$
$$-Im_{\eta\zeta}[\sin(\theta_G+\phi_k)][\sin(\theta_G+\phi_k)\{v_k'\ddot{\phi}_k+2\dot{v}'_k\dot{\phi}_k-\ddot{w}_k'+w_k'\dot{\phi}_k^2$$
$$-\cos\theta_I(<\ddot{\theta}_x+\dot{\theta}_y>\sin\psi_k+<\dot{\theta}_x-\ddot{\theta}_y>\cos\psi_k)-\dot{\phi}_k\dot{\theta}_z\cos\theta_I$$
$$+\ddot{\theta}_I(<\beta_s\cos\theta_I-\beta_d\sin\theta_I>-1)+\dot{\theta}_I(-\dot{\theta}_I<\beta_s\sin\theta_I-\beta_d\cos\theta_I>$$

$$+\dot{\phi}_k < -1 + \beta_d \cos\theta_I + \beta_s \sin\theta_I >) - \dot{\phi}_k \cos\theta_I\} + \cos(\theta_G + \phi_k)\{-\ddot{v}'_k + v_k\dot{\phi}_k^2$$

$$-2\dot{w}'_k\dot{\phi}_k - w'_k\ddot{\phi}_k - \dot{\phi}_k \cos\theta_I < \dot{\theta}_x \sin\psi_k - \dot{\theta}_y \cos\psi_k > + \ddot{\theta}_z \cos\theta_I$$

$$+\ddot{\theta}_I(1 - < \beta_d \cos\theta_I + \beta_s \sin\theta_I >)$$

$$+\dot{\theta}_I(\dot{\phi}_k < \beta_s \cos\theta_I - \beta_d \sin\theta_I > + \dot{\theta}_I < \beta_d \sin\theta_I - \beta_s \cos\theta_I > - \dot{\phi}_k) + \dot{\phi}_k \sin\theta_I\}]$$

$$-m\eta_m[-\cos(\theta_G + \phi_k)][\ddot{R}_x\{\cos\psi_k - \sin\psi_k(-\beta_s \cos\theta_I + \beta_d \sin\theta_I)\}$$

$$+\dot{R}_x\{-\sin\psi_k - \cos\psi_k(-\beta_s \cos\theta_I + \beta_d \sin\theta_I)$$

$$-\dot{\theta}_I \sin\psi_k(\beta_s \sin\theta_I + \beta_d \cos\theta_I)\} + \ddot{R}_y\{\sin\psi_k + \cos\psi_k(-\beta_s \cos\theta_I + \beta_d \sin\theta_I)\}$$

$$+\dot{R}_y\{\cos\psi_k - \sin\psi_k(-\beta_s \cos\theta_I + \beta_d \sin\theta_I + \dot{\theta}_I \cos\psi_k(\beta_s \sin\theta_I + \beta_d \cos\theta_I)\}$$

$$+\ddot{R}_z(\beta_p + \beta_d \cos\theta_I + \beta_s \sin\theta_I) + \dot{R}_z\dot{\theta}_I(-\beta_d \sin\theta_I + \beta_s \cos\theta_I)$$

$$+\{a\ddot{\theta}_I(\beta_s \sin\theta_I + \beta_d \cos\theta_I) + a\dot{\theta}_I^2(\beta_s \cos\theta_I - \beta_d \sin\theta_I) + \ddot{u}_k\}$$

$$-w_k \cos\theta_I\{(\ddot{\theta}_x + \dot{\theta}_y)\sin\psi_k + (\dot{\theta}_x - \ddot{\theta}_y)\cos\psi_k\} - \dot{w}_k \cos\theta_I(\dot{\theta}_x \sin\psi_k - \dot{\theta}_y \cos\psi_k)$$

$$-\ddot{\theta}_z \cos\theta_I(a\cos\theta_I + v_k) - \dot{\theta}_z\dot{v}_k - \ddot{\theta}_I\{(-w_k + e_1 + e_2)(\beta_s \cos\theta_I - \beta_d \sin\theta_I)$$

$$+(v_k + a\cos\theta_I) + (e_1 + e_2 - v_k - a\cos\theta_I)(\beta_d \cos\theta_I + \beta_s \sin\theta_I) + w_k\}$$

$$-\dot{\theta}_I\{-\dot{w}_k(\beta_s \cos\theta_I - \beta_d \sin\theta_I) - \dot{\theta}_I(-w_k + e_1 + e_2)(\beta_s \sin\theta_I - \beta_d \cos\theta_I)$$

$$+\dot{v}_k(1 - \beta_d \cos\theta_I - \beta_s \sin\theta_I) + \dot{\theta}_I(e_1 + e_2 - a\cos\theta_I - v_k)(-\beta_d \sin\theta_I + \beta_s \cos\theta_I) + \dot{w}_k\}$$

$$-\dot{\theta}_I \cos\theta_I(e_1 + e_2)(\beta_s \sin\theta_I + \beta_d \cos\theta_I) + \dot{v}_k$$

$$-\{(e_1 + e_2)(-\beta_d \sin\theta_I + \beta_s \cos\theta_I) + \dot{w}_k\}\dot{\theta}_I \sin\theta_I]$$

$$-m\zeta_m[\sin(\theta_G + \phi_k)][\ddot{R}_x\{\cos\psi_k - \sin\psi_k(-\beta_s \cos\theta_I + \beta_d \sin\theta_I)\}$$

$$+\dot{R}_x\{-\sin\psi_k - \cos\psi_k(-\beta_s \cos\theta_I + \beta_d \sin\theta_I)$$

$$-\dot{\theta}_I \sin\psi_k(\beta_s \sin\theta_I + \beta_d \cos\theta_I)\} + \ddot{R}_y\{\sin\psi_k + \cos\psi_k(-\beta_s \cos\theta_I + \beta_d \sin\theta_I)\}$$

$$+\dot{R}_y\{\cos\psi_k - \sin\psi_k(-\beta_s \cos\theta_I + \beta_d \sin\theta_I + \dot{\theta}_I \cos\psi_k(\beta_s \sin\theta_I + \beta_d \cos\theta_I)\}$$

$$+\ddot{R}_z(\beta_p + \beta_d \cos\theta_I + \beta_s \sin\theta_I) + \dot{R}_z\dot{\theta}_I(-\beta_d \sin\theta_I + \beta_s \cos\theta_I)$$

$$+\{a\ddot{\theta}_I(\beta_s \sin\theta_I + \beta_d \cos\theta_I) + a\dot{\theta}_I^2(\beta_s \cos\theta_I - \beta_d \sin\theta_I) + \ddot{u}_k\}$$

$$-w_k \cos\theta_I\{(\ddot{\theta}_x + \dot{\theta}_y)\sin\psi_k + (\dot{\theta}_x - \ddot{\theta}_y)\cos\psi_k\} - \dot{w}_k \cos\theta_I(\dot{\theta}_x \sin\psi_k - \dot{\theta}_y \cos\psi_k)$$

$$
\begin{aligned}
&-\ddot{\theta}_z \cos\theta_I (a\cos\theta_I + v_k) - \dot{\theta}_z \dot{v}_k - \ddot{\theta}_I \{(-w_k + e_1 + e_2)(\beta_s \cos\theta_I - \beta_d \sin\theta_I) \\
&+(v_k + a\cos\theta_I) + (e_1 + e_2 - v_k - a\cos\theta_I)(\beta_d \cos\theta_I + \beta_s \sin\theta_I) + w_k\} \\
&-\dot{\theta}_I \{-\dot{w}_k(\beta_s \cos\theta_I - \beta_d \sin\theta_I) - \dot{\theta}_I(-w_k + e_1 + e_2)(\beta_s \sin\theta_I - \beta_d \cos\theta_I) \\
&+\dot{v}_k(1 - \beta_d \cos\theta_I - \beta_s \sin\theta_I) + \dot{\theta}_I(e_1 + e_2 - a\cos\theta_I - v_k)(-\beta_d \sin\theta_I + \beta_s \cos\theta_I) + \dot{w}_k\} \\
&-\dot{\theta}_I \cos\theta_I (e_1 + e_2)(\beta_s \sin\theta_I + \beta_d \cos\theta_I) + \dot{v}_k \\
&-\{(e_1 + e_2)(-\beta_d \sin\theta_I + \beta_s \cos\theta_I) + \dot{w}_k\}\dot{\theta}_I \sin\theta_I]
\end{aligned}
$$

$$
\begin{aligned}
&\bar{Z}'_{w} = \\
&Im_{\zeta\zeta}[\{-\dot{\phi}_k \cos(\theta_G + \phi_k)\}\{\sin(\theta_G + \phi_k)(\dot{w}'_k + \dot{\theta}_x \sin\psi_k \cos\theta_I \\
&-\dot{\theta}_y \cos\psi_k \cos\theta_I - \dot{\theta}_I < \beta_s \cos\theta_I - \beta_d \sin\theta_I - 1 > + \sin\theta_I) \\
&+\cos(\theta_G + \phi_k)(-\dot{v}'_k - \dot{\theta}_z - \dot{\theta}_I < 1 - \beta_d \cos\theta_I + \beta_s \sin\theta_I > - \cos\theta_I\}] \\
&+Im_{\eta\eta}[\{\dot{\phi}_k \sin(\theta_G + \phi_k)\}\{\sin(\theta_G + \phi_k)(\dot{v}'_k - \dot{\theta}_z \\
&-\dot{\theta}_I < 1 - \beta_d \cos\theta_I - \beta_s \sin\theta_I > - \cos\theta_I) \\
&+\cos(\theta_G + \phi_k)(-\dot{w}'_k + \dot{\theta}_x \sin\psi_k - \dot{\theta}_y \cos\psi_k + \dot{\theta}_I < \beta_s \cos\theta_I - \beta_d \sin\theta_I + 1 > - \sin\theta_I)\}] \\
&+Im_{\eta\zeta}[\{-\dot{\phi}_k \cos(\theta_G + \phi_k)\}\{\sin(\theta_G + \phi_k)(\dot{v}'_k - \dot{\theta}_z \\
&-\dot{\theta}_I < 1 - \beta_d \cos\theta_I - \beta_s \sin\theta_I > - \cos\theta_I) \\
&+\cos(\theta_G + \phi_k)(-\dot{w}'_k + \dot{\theta}_x \sin\psi_k - \dot{\theta}_y \cos\psi_k + \dot{\theta}_I < \beta_s \cos\theta_I - \beta_d \sin\theta_I + 1 > - \sin\theta_I)\}] \\
&+\{\dot{\phi}_k \sin(\theta_G + \phi_k)\}\{\sin(\theta_G + \phi_k)(\dot{w}'_k + \dot{\theta}_x \sin\psi_k \cos\theta_I - \dot{\theta}_y \cos\psi_k \cos\theta_I \\
&-\dot{\theta}_I < \beta_s \cos\theta_I - \beta_d \sin\theta_I - 1 > + \sin\theta_I) \\
&+\cos(\theta_G + \phi_k)(-\dot{v}'_k - \dot{\theta}_z - \dot{\theta}_I < 1 - \beta_d \cos\theta_I + \beta_s \sin\theta_I > - \cos\theta_I\}] \\
&+m\eta_m[[\{-\dot{\phi}_k \cos(\theta_G + \phi_k)\}\{\dot{R}_x \cos\psi_k + \dot{R}_y \sin\psi_k \\
&-\cos\theta_I(a\cos\theta_I + v_k)(1 + \dot{\theta}_I) - w_k \sin\theta_I\}] \\
&+m\zeta_m[\{\dot{\phi}_k \sin(\theta_G + \phi_k)\}\{\dot{R}_x \cos\psi_k + \dot{R}_y \sin\psi_k \\
&-\cos\theta_I(a\cos\theta_I + v_k)(1 + \dot{\theta}_I) - w_k \sin\theta_I\}] \\
&+Im_{\zeta\zeta}[\sin(\theta_G + \phi_k)\{-\dot{\theta}_z + \dot{\theta}_I(\beta_d \cos\theta_I + \beta_s \sin\theta_I - v'_k + 1) + \cos\theta_I\}] \\
&[\sin(\theta_G + \phi_k)\{-\dot{\phi}_k + w'_k) + \dot{\theta}_I(\beta_d < \cos\theta_I + \sin\theta_I > + \beta_s < \sin\theta_I - \cos\theta_I > + 1) \\
&+(\beta_p + \beta_d \cos\theta_I + \beta_s \sin\theta_I)\} + v'_k \cos(\theta_G + \phi_k)(\dot{\theta}_I \cos\theta_I - 1)] \\
&+Im_{\eta\eta}[\cos(\theta_G + \phi_k)\{\dot{\theta}_z + \dot{\theta}_I(\beta_d \cos\theta_I + \beta_s \sin\theta_I + v'_k + 1) + \cos\theta_I\}] \\
&[\sin(\theta_G + \phi_k)\{v'_k(1 - \dot{\theta}_I \cos\theta_I)\} + \cos(\theta_G + \phi_k)\{(-\dot{\phi}_k + w'_k) \\
&-\dot{\theta}_I(\beta_d < \cos\theta_I + \sin\theta_I > + \beta_s < \sin\theta_I - \cos\theta_I > + 1) - (\beta_p + \beta_d \cos\theta_I + \beta_s \sin\theta_I)\}]
\end{aligned}
$$

$$+Im_{\eta\zeta}[\sin(\theta_G + \phi_k)\{-\dot{\theta}_z + \dot{\theta}_I(\beta_d \cos\theta_I + \beta_s \sin\theta_I - v_k' + 1) + \cos\theta_I\}]$$

$$[\sin(\theta_G + \phi_k)\{v_k'(1 - \dot{\theta}_I \cos\theta_I)\} + \cos(\theta_G + \phi_k)\{(-\dot{\phi}_k + w_k')$$

$$-\dot{\theta}_I(\beta_d < \cos\theta_I + \sin\theta_I > +\beta_s < \sin\theta_I - \cos\theta_I > +1) - (\beta_p + \beta_d \cos\theta_I + \beta_s \sin\theta_I)\}]$$

$$+Im_{\eta\zeta}[\cos(\theta_G + \phi_k)\{\dot{\theta}_z + \dot{\theta}_I(\beta_d \cos\theta_I + \beta_s \sin\theta_I + v_k' + 1) + \cos\theta_I\}]$$

$$[\sin(\theta_G + \phi_k)\{-\dot{\phi}_k + w_k') + \dot{\theta}_I(\beta_d < \cos\theta_I + \sin\theta_I > +\beta_s < \sin\theta_I - \cos\theta_I > +1)$$

$$+(\beta_p + \beta_d \cos\theta_I + \beta_s \sin\theta_I)\} + v_k' \cos(\theta_G + \phi_k)(\dot{\theta}_I \cos\theta_I - 1)]$$

$$+m\eta_m[\sin(\theta_G + \phi_k)\{-\dot{\theta}_z + \dot{\theta}_I(\beta_d \cos\theta_I + \beta_s \sin\theta_I - v_k' + 1) + \cos\theta_I\}]$$

$$[-\dot{\theta}_I w_k - (e_1 + e_2 + < n - 1 > l_e + x_k)(1 - \dot{\theta}_I)]$$

$$+m\zeta_m[\cos(\theta_G + \phi_k)\{\dot{\theta}_z + \dot{\theta}_I(\beta_d \cos\theta_I + \beta_s \sin\theta_I + v_k' + 1) + \cos\theta_I\}]$$

$$[-\dot{\theta}_I w_k - (e_1 + e_2 + < n - 1 > l_e + x_k)(1 - \dot{\theta}_I)]$$

$$+Im_{\zeta\zeta}[\{\sin(\theta_G + \phi_k)(\dot{\theta}_I < \beta_d \cos\theta_I + \beta_s \sin\theta_I + 1 - v_k' > + \cos\theta_I)\}$$

$$\{\sin(\theta_G + \phi_k)(\dot{\theta}_x \cos\psi_k + \dot{\theta}_y \sin\psi_k)\}]$$

$$+Im_{\eta\eta}[\{\cos(\theta_G + \phi_k)(\dot{\theta}_I < \beta_d \cos\theta_I + \beta_s \sin\theta_I + 1 + v_k' > + \cos\theta_I)\}$$

$$\{-\cos(\theta_G + \phi_k)(\dot{\theta}_x \cos\psi_k + \dot{\theta}_y \sin\psi_k)\}]$$

$$+Im_{\eta\zeta}[\{\sin(\theta_G + \phi_k)(\dot{\theta}_I < \beta_d \cos\theta_I + \beta_s \sin\theta_I + 1 - v_k' > + \cos\theta_I)\}$$

$$\{-\cos(\theta_G + \phi_k)(\dot{\theta}_x \cos\psi_k + \dot{\theta}_y \sin\psi_k)\}$$

$$+\{\cos(\theta_G + \phi_k)(\dot{\theta}_I < \beta_d \cos\theta_I + \beta_s \sin\theta_I + 1 + v_k' > + \cos\theta_I)\}\{\sin(\theta_G + \phi_k)$$

$$(\dot{\theta}_x \cos\psi_k + \dot{\theta}_y \sin\psi_k)\}]$$

$$+m\eta_m[\{\sin(\theta_G + \phi_k)(\dot{\theta}_I < \beta_d \cos\theta_I + \beta_s \sin\theta_I + 1 - v_k' > +1)\}$$

$$\{-\dot{R}_x \sin\psi_k \cos\theta_I + \dot{R}_y \cos\psi_k \cos\theta_I + (\dot{\theta}_z \cos\theta_I + \beta_d \cos\theta_I + \beta_s \sin\theta_I)$$

$$(< n - 1 > l_e + x_k)\}]$$

$$+m\zeta_m[\{\cos(\theta_G + \phi_k)(\dot{\theta}_I < \beta_d \cos\theta_I + \beta_s \sin\theta_I + 1 + v_k' > +1)\}$$

$$\{-\dot{R}_x \sin\psi_k \cos\theta_I + \dot{R}_y \cos\psi_k \cos\theta_I + (\dot{\theta}_z \cos\theta_I + \beta_d \cos\theta_I + \beta_s \sin\theta_I)$$

$$(< n - 1 > l_e + x_k)\}]$$

$$+Im_{\zeta\zeta}[\sin(\theta_G+\phi_k)(1+\dot{\theta}_I)][\sin(\theta_G+\phi_k)\{-\dot{\theta}_z w_k'\cos\theta_I+\dot{\theta}_I w_k'(\beta_d\cos\theta_I+\beta_s\sin\theta_I)\}$$

$$+\cos(\theta_G+\phi_k)\{-\dot{\theta}_z v_k'\cos\theta_I+\dot{\theta}_I v_k'(\beta_d\cos\theta_I+\beta_s\sin\theta_I)-\dot{\theta}_I w_k' v_k'\}]$$

$$+Im_{\eta\eta}[\cos(\theta_G+\phi_k)(1+\dot{\theta}_I)][\sin(\theta_G+\phi_k)\{\dot{\theta}_z v_k'\cos\theta_I-\dot{\theta}_I v_k'(\beta_d\cos\theta_I+\beta_s\sin\theta_I)$$

$$+\dot{\theta}_I w_k' v_k'\}+\cos(\theta_G+\phi_k)\{\dot{\theta}_z w_k'\cos\theta_I+\dot{\theta}_I w_k'(\beta_d\cos\theta_I+\beta_s\sin\theta_I)\}]$$

$$+Im_{\eta\zeta}[\sin(\theta_G+\phi_k)(1+\dot{\theta}_I)][\sin(\theta_G+\phi_k)\{\dot{\theta}_z v_k'\cos\theta_I-\dot{\theta}_I v_k'(\beta_d\cos\theta_I+\beta_s\sin\theta_I)$$

$$+\dot{\theta}_I w_k' v_k'\}+\cos(\theta_G+\phi_k)\{\dot{\theta}_z w_k'\cos\theta_I+\dot{\theta}_I w_k'(\beta_d\cos\theta_I+\beta_s\sin\theta_I)\}]$$

$$+Im_{\eta\zeta}[\cos(\theta_G+\phi_k)(1+\dot{\theta}_I)][\sin(\theta_G+\phi_k)\{-\dot{\theta}_z w_k'\cos\theta_I+\dot{\theta}_I w_k'(\beta_d\cos\theta_I+\beta_s\sin\theta_I)\}$$

$$+\cos(\theta_G+\phi_k)\{-\dot{\theta}_z v_k'\cos\theta_I+\dot{\theta}_I v_k'(\beta_d\cos\theta_I+\beta_s\sin\theta_I)-\dot{\theta}_I w_k' v_k'\}]$$

$$+m\eta_m[\sin(\theta_G+\phi_k)(1+\dot{\theta}_I)][(\dot{R}_x\cos\psi_k+\dot{R}_y\sin\psi_k)(\beta_s\cos\theta_I-\beta_d\sin\theta_I)$$

$$-\dot{\theta}_I(\beta_s\sin\theta_I+\beta_d\cos\theta_I)(e_1+e_2+2a\sin\theta_I)-w_k(\dot{\theta}_x\cos\psi_k+\dot{\theta}_y\sin\psi_k)$$

$$+\dot{\theta}_z\cos\theta_I(e_1+e_2)$$

$$+\dot{\theta}_I(a\cos\theta_I<\beta_s\cos\theta_I-\beta_d\sin\theta_I>-u_k)-(\beta_p+\beta_d\cos\theta_I+\beta_s\sin\theta_I)w_k$$

$$+a\cos\theta_I(\beta_s\cos\theta_I-\beta_d\sin\theta_I)]$$

$$+m\zeta_m[\cos(\theta_G+\phi_k)(1+\dot{\theta}_I)][(\dot{R}_x\cos\psi_k+\dot{R}_y\sin\psi_k)(\beta_s\cos\theta_I-\beta_d\sin\theta_I)$$

$$-\dot{\theta}_I(\beta_s\sin\theta_I+\beta_d\cos\theta_I)(e_1+e_2+2a\sin\theta_I)-w_k(\dot{\theta}_x\cos\psi_k+\dot{\theta}_y\sin\psi_k)$$

$$+\dot{\theta}_z\cos\theta_I(e_1+e_2)$$

$$+\dot{\theta}_I(a\cos\theta_I<\beta_s\cos\theta_I-\beta_d\sin\theta_I>-u_k)-(\beta_p+\beta_d\cos\theta_I+\beta_s\sin\theta_I)w_k$$

$$+a\cos\theta_I(\beta_s\cos\theta_I-\beta_d\sin\theta_I)]$$

$$+Im_{\zeta\zeta}[\sin(\theta_G+\phi_k)(\cos\theta_I+\sin\theta_I)\{-\dot{\theta}_x\sin\psi_k+\dot{\theta}_y\cos\psi_k+\dot{\theta}_I(\beta_s\cos\theta_I-\beta_d\sin\theta_I)\}]$$

$$[\dot{\theta}_I w_k'\sin(\theta_G+\phi_k)+\cos(\theta_G+\phi_k)\{-\dot{\phi}_k+\dot{\theta}_I(\beta_d<\cos\theta_I+\sin\theta_I>$$

$$+\beta_s<\sin\theta_I-\cos\theta_I>+1)+\beta_p+\beta_d\cos\theta_I+\beta_s\sin\theta_I+\dot{\theta}_I v_k'\}]$$

$$+Im_{\eta\eta}[\cos(\theta_G+\phi_k)(\cos\theta_I+\sin\theta_I)\{-\dot{\theta}_x\sin\psi_k+\dot{\theta}_y\cos\psi_k+\dot{\theta}_I(\beta_s\cos\theta_I-\beta_d\sin\theta_I)\}]$$

$$[\sin(\theta_G+\phi_k)\{-\dot{\phi}_k+\dot{\theta}_I(\beta_d<\cos\theta_I+\sin\theta_I>+\beta_s<\sin\theta_I-\cos\theta_I>+1)$$

$$+\beta_p+\beta_d\cos\theta_I+\beta_s\sin\theta_I-\dot{\theta}_I v_k'\}+\dot{\theta}_I w_k'\cos(\theta_G+\phi_k)]$$

$$+Im_{\eta\zeta}[\sin(\theta_G+\phi_k)(\cos\theta_I+\sin\theta_I)\{-\dot{\theta}_x\sin\psi_k+\dot{\theta}_y\cos\psi_k+\dot{\theta}_I(\beta_s\cos\theta_I-\beta_d\sin\theta_I)\}]$$

$$[\sin(\theta_G+\phi_k)\{-\dot{\phi}_k+\dot{\theta}_I(\beta_d<\cos\theta_I+\sin\theta_I>+\beta_s<\sin\theta_I-\cos\theta_I>+1)$$

$$+\beta_p+\beta_d\cos\theta_I+\beta_s\sin\theta_I-\dot{\theta}_I v'_k\}+\dot{\theta}_I w'_k\cos(\theta_G+\phi_k)]$$

$$+Im_{\eta\zeta}[\cos(\theta_G+\phi_k)(\cos\theta_I+\sin\theta_I)\{-\dot{\theta}_x\sin\psi_k+\dot{\theta}_y\cos\psi_k+\dot{\theta}_I(\beta_s\cos\theta_I-\beta_d\sin\theta_I)\}]$$

$$[\dot{\theta}_I w'_k\sin(\theta_G+\phi_k)+\cos(\theta_G+\phi_k)\{-\dot{\phi}_k+\dot{\theta}_I(\beta_d<\cos\theta_I+\sin\theta_I>$$

$$+\beta_s<\sin\theta_I-\cos\theta_I>+1)+\beta_p+\beta_d\cos\theta_I+\beta_s\sin\theta_I+\dot{\theta}_I v'_k\}]$$

$$+m\eta_m[\sin(\theta_G+\phi_k)(\cos\theta_I+\sin\theta_I)\{-\dot{\theta}_x\sin\psi_k+\dot{\theta}_y\cos\psi_k+\dot{\theta}_I(\beta_s\cos\theta_I-\beta_d\sin\theta_I)\}]$$

$$[\dot{w}_k+a\cos\theta_I+v_k-(\beta_s\cos\theta_I-\beta_d\sin\theta_I+1)(<n-1>l_e+x_k)-(e_1+e_2)]$$

$$+m\zeta_m[\cos(\theta_G+\phi_k)(\cos\theta_I+\sin\theta_I)\{-\dot{\theta}_x\sin\psi_k+\dot{\theta}_y\cos\psi_k+\dot{\theta}_I(\beta_s\cos\theta_I-\beta_d\sin\theta_I)\}]$$

$$[\dot{w}_k+a\cos\theta_I+v_k-(\beta_s\cos\theta_I-\beta_d\sin\theta_I+1)(<n-1>l_e+x_k)-(e_1+e_2)]$$

$$+Im_{\zeta\zeta}[\{\dot{\theta}_I\sin(\theta_G+\phi_k)(\cos\theta_I+\sin\theta_I)(\beta_s\cos\theta_I-\beta_d\sin\theta_I)\}$$

$$\{\cos(\theta_G+\phi_k)\cos\theta_I(\dot{\theta}_x\cos\psi_k+\dot{\theta}_y\sin\psi_k)\}]$$

$$+Im_{\eta\eta}[\{\dot{\theta}_I\cos(\theta_G+\phi_k)(\cos\theta_I+\sin\theta_I)(\beta_s\cos\theta_I-\beta_d\sin\theta_I)\}$$

$$\{\sin(\theta_G+\phi_k)\cos\theta_I(\dot{\theta}_x\cos\psi_k+\dot{\theta}_y\sin\psi_k)\}]$$

$$+Im_{\eta\zeta}[\{\dot{\theta}_I\sin(\theta_G+\phi_k)(\cos\theta_I+\sin\theta_I)(\beta_s\cos\theta_I-\beta_d\sin\theta_I)\}$$

$$\{\sin(\theta_G+\phi_k)\cos\theta_I(\dot{\theta}_x\cos\psi_k+\dot{\theta}_y\sin\psi_k)\}$$

$$+\{\dot{\theta}_I\cos(\theta_G+\phi_k)(\cos\theta_I+\sin\theta_I)(\beta_s\cos\theta_I-\beta_d\sin\theta_I)\}$$

$$\{\cos(\theta_G+\phi_k)\cos\theta_I(\dot{\theta}_x\cos\psi_k+\dot{\theta}_y\sin\psi_k)\}]$$

$$+m\eta_m[\{\dot{\theta}_I\sin(\theta_G+\phi_k)(\cos\theta_I+\sin\theta_I)(\beta_s\cos\theta_I-\beta_d\sin\theta_I)\}$$

$$\{\dot{R}_z\cos\theta_I+(\dot{\theta}_x\sin\psi_k-\dot{\theta}_y\cos\psi_k)(<n-1>l_e+x_k)\}]$$

$$+m\zeta_m[\{\dot{\theta}_I\cos(\theta_G+\phi_k)(\cos\theta_I+\sin\theta_I)(\beta_s\cos\theta_I-\beta_d\sin\theta_I)\}$$

$$\{\dot{R}_z\cos\theta_I+(\dot{\theta}_x\sin\psi_k-\dot{\theta}_y\cos\psi_k)(<n-1>l_e+x_k)\}]$$

$$-Im_{\zeta\zeta}[\{-\dot{\phi}_k\cos(\theta_G+\phi_k)\}\{\sin(\theta_G+\phi_k)(v'_k\dot{\phi}_k-\dot{w}'_k+\dot{\theta}_x\sin\psi_k\cos\theta_I$$

$$-\dot{\theta}_y\cos\psi_k\cos\theta_I-\dot{\theta}_z\sin\theta_I)-\dot{\theta}_I<\beta_s\cos\theta_I-\beta_d\sin\theta_I>+\dot{\theta}_I+\sin\theta_I)$$

$$+\cos(\theta_G+\phi_k)(-\dot{v'}_k-w_k'\dot{\phi}_k-\dot{\theta}_z\cos\theta_I-\dot{\theta}_I<1-\beta_d\cos\theta_I-\beta_s\sin\theta_I>-\cos\theta_I)\}]$$

$$-Im_{\eta\eta}[\{\dot{\phi}_k\sin(\theta_G+\phi_k)\}\{\sin(\theta_G+\phi_k)(\dot{v'}_k+w_k'\dot{\phi}_k-\dot{\theta}_z\cos\theta_I-\dot{\theta}_I<1-\beta_d\cos\theta_I$$

$$-\beta_s\sin\theta_I>-\cos\theta_I)+\cos(\theta_G+\phi_k)(v_k'\dot{\phi}_k-\dot{w}_k'-\dot{\theta}_x\sin\psi_k\cos\theta_I+\dot{\theta}_y\cos\psi_k\cos\theta_I$$

$$+\dot{\theta}_z\sin\theta_I)-\dot{\theta}_I<-\beta_s\cos\theta_I+\beta_d\sin\theta_I+\dot{\theta}_I>-\sin\theta_I)\}]$$

$$+Im_{\eta\zeta}[\{-\dot{\phi}_k\cos(\theta_G+\phi_k)\}\{\sin(\theta_G+\phi_k)(\dot{v'}_k+w_k'\dot{\phi}_k-\dot{\theta}_z\cos\theta_I-\dot{\theta}_I<1-\beta_d\cos\theta_I$$

$$-\beta_s\sin\theta_I>-\cos\theta_I)+\cos(\theta_G+\phi_k)(v_k'\dot{\phi}_k-\dot{w}_k'-\dot{\theta}_x\sin\psi_k\cos\theta_I+\dot{\theta}_y\cos\psi_k\cos\theta_I$$

$$+\dot{\theta}_z\sin\theta_I)-\dot{\theta}_I<-\beta_s\cos\theta_I+\beta_d\sin\theta_I+\dot{\theta}_I>-\sin\theta_I)\}]$$

$$+\{\dot{\phi}_k\sin(\theta_G+\phi_k)\}\{\sin(\theta_G+\phi_k)(v_k'\dot{\phi}_k-\dot{w}_k'+\dot{\theta}_x\sin\psi_k\cos\theta_I-\dot{\theta}_y\cos\psi_k\cos\theta_I$$

$$-\dot{\theta}_z\sin\theta_I)-\dot{\theta}_I<\beta_s\cos\theta_I-\beta_d\sin\theta_I>+\dot{\theta}_I+\sin\theta_I)$$

$$+\cos(\theta_G+\phi_k)(-\dot{v'}_k-w_k'\dot{\phi}_k-\dot{\theta}_z\cos\theta_I-\dot{\theta}_I<1-\beta_d\cos\theta_I-\beta_s\sin\theta_I>-\cos\theta_I)\}]$$

$$-m\eta_m[\{-\dot{\phi}_k\cos(\theta_G+\phi_k)\}\{\dot{R}_x(\cos\psi_k-\sin\psi_k<-\beta_s\cos\theta_I+\beta_d\sin\theta_I>)$$

$$+\dot{R}_y(\sin\psi_k+\cos\psi_k<-\beta_s\cos\theta_I+\beta_d\sin\theta_I>)$$

$$+\dot{R}_z(\beta_p+\beta_d\cos\theta_I+\beta_s\sin\theta_I)+a\dot{\theta}_I(\beta_s\sin\theta_I+\beta_d\cos\theta_I)+\dot{u}_k-\dot{\theta}_xw_k\sin\psi_k\cos\theta_I$$

$$+\dot{\theta}_yw_k\cos\psi_k\cos\theta_I-\dot{\theta}_z\cos\theta_I(a\cos\theta_I+v_k)$$

$$-\dot{\theta}_I(<-w_k+e_1+e_2><\beta_s\cos\theta_I-\beta_d\sin\theta_I>$$

$$+<v_k+a\cos\theta_I)><1-\beta_d\cos\theta_I-\beta_s\sin\theta_I>$$

$$+<e_1+e_2><\beta_d\cos\theta_I+\beta_s\sin\theta_I>$$

$$+w_k)-\cos\theta_I(a\cos\theta_I+<e_1+e_2><\beta_s\cos\theta_I-\beta_d\sin\theta_I>+v_k)-\sin\theta_I)$$

$$(<e_1+e_2><\beta_d\cos\theta_I+\beta_s\sin\theta_I>+w_k)\}]$$

$$-m\zeta_m[\{\dot{\phi}_k\sin(\theta_G+\phi_k)\}\{\dot{R}_x(\cos\psi_k-\sin\psi_k<-\beta_s\cos\theta_I+\beta_d\sin\theta_I>)$$

$$+\dot{R}_y(\sin\psi_k+\cos\psi_k<-\beta_s\cos\theta_I+\beta_d\sin\theta_I>)$$

$$+\dot{R}_z(\beta_p+\beta_d\cos\theta_I+\beta_s\sin\theta_I)+a\dot{\theta}_I(\beta_s\sin\theta_I+\beta_d\cos\theta_I)+\dot{u}_k-\dot{\theta}_xw_k\sin\psi_k\cos\theta_I$$

$$+\dot{\theta}_yw_k\cos\psi_k\cos\theta_I-\dot{\theta}_z\cos\theta_I(a\cos\theta_I+v_k)$$

$$+w_k)-\cos\theta_I(a\cos\theta_I+<e_1+e_2><\beta_s\cos\theta_I-\beta_d\sin\theta_I>+v_k)$$

$$-\sin\theta_I)(<e_1+e_2><\beta_d\cos\theta_I+\beta_s\sin\theta_I>+w_k)\}]$$

$$-Im_{\zeta\zeta}[-\sin(\theta_G+\phi_k)][\sin(\theta_G+\phi_k)\{v_k'\ddot{\phi}_k+2\dot{v}'_k\dot{\phi}_k-\ddot{w}_k'+w_k'\dot{\phi}_k^2$$

$$-\cos\theta_I(<\ddot{\theta}_x+\dot{\theta}_y>\sin\psi_k+<\dot{\theta}_x-\ddot{\theta}_y>\cos\psi_k)-\dot{\phi}_k\dot{\theta}_z\cos\theta_I$$

$$+\ddot{\theta}_I(<\beta_s\cos\theta_I-\beta_d\sin\theta_I>-1)+\dot{\theta}_I(-\dot{\theta}_I<\beta_s\sin\theta_I-\beta_d\cos\theta_I>$$

$$+\dot{\phi}_k<-1+\beta_d\cos\theta_I+\beta_s\sin\theta_I>)-\dot{\phi}_k\cos\theta_I\}+\cos(\theta_G+\phi_k)\{-\ddot{v}_k'+v_k\dot{\phi}_k^2$$

$$-2\dot{w}_k'\dot{\phi}_k-w_k'\ddot{\phi}_k-\dot{\phi}_k\cos\theta_I<\dot{\theta}_x\sin\psi_k-\dot{\theta}_y\cos\psi_k>+\ddot{\theta}_z\cos\theta_I$$

$$+\ddot{\theta}_I(1-<\beta_d\cos\theta_I+\beta_s\sin\theta_I>)$$

$$+\dot{\theta}_I(\dot{\phi}_k<\beta_s\cos\theta_I-\beta_d\sin\theta_I>+\dot{\theta}_I<\beta_d\sin\theta_I-\beta_s\cos\theta_I>-\dot{\phi}_k)+\dot{\phi}_k\sin\theta_I\}]$$

$$-Im_{\eta\eta}[-\cos(\theta_G+\phi_k)][\sin(\theta_G+\phi_k)\{\ddot{v}_k'-v_k\dot{\phi}_k^2+2\dot{w}_k'\dot{\phi}_k+w_k'\ddot{\phi}_k$$

$$-\dot{\phi}_k\cos\theta_I<\dot{\theta}_x\sin\psi_k-\dot{\theta}_y\cos\psi_k>+\ddot{\theta}_z\cos\theta_I+\ddot{\theta}_I(1-<\beta_d\cos\theta_I+\beta_s\sin\theta_I>)$$

$$+\dot{\theta}_I(\dot{\phi}_k<\beta_s\cos\theta_I-\beta_d\sin\theta_I>+\dot{\theta}_I<\beta_d\sin\theta_I-\beta_s\cos\theta_I>-\dot{\phi}_k)+\dot{\phi}_k\sin\theta_I\}$$

$$+\cos(\theta_G+\phi_k)\{v_k'\ddot{\phi}_k+2\dot{v}'_k\dot{\phi}_k-\ddot{w}_k'+w_k'\dot{\phi}_k^2$$

$$+\cos\theta_I(<\ddot{\theta}_x+\dot{\theta}_y>\sin\psi_k+<\dot{\theta}_x-\ddot{\theta}_y>\cos\psi_k)+\dot{\phi}_k\dot{\theta}_z\cos\theta_I$$

$$+\ddot{\theta}_I(-\beta_s\cos\theta_I+\beta_d\sin\theta_I+1)+\dot{\theta}_I(\dot{\theta}_I<\beta_s\sin\theta_I-\beta_d\cos\theta_I>$$

$$+\dot{\phi}_k<1-\beta_d\cos\theta_I+\beta_s\sin\theta_I>)+\dot{\phi}_k\cos\theta_I\}]$$

$$-Im_{\eta\zeta}[-\sin(\theta_G+\phi_k)][\sin(\theta_G+\phi_k)\{\ddot{v}_k'-v_k\dot{\phi}_k^2+2\dot{w}_k'\dot{\phi}_k+w_k'\ddot{\phi}_k$$

$$-\dot{\phi}_k\cos\theta_I<\dot{\theta}_x\sin\psi_k-\dot{\theta}_y\cos\psi_k>+\ddot{\theta}_z\cos\theta_I+\ddot{\theta}_I(1-<\beta_d\cos\theta_I+\beta_s\sin\theta_I>)$$

$$+\dot{\theta}_I(\dot{\phi}_k<\beta_s\cos\theta_I-\beta_d\sin\theta_I>+\dot{\theta}_I<\beta_d\sin\theta_I-\beta_s\cos\theta_I>-\dot{\phi}_k)+\dot{\phi}_k\sin\theta_I\}$$

$$+\cos(\theta_G+\phi_k)\{v_k'\ddot{\phi}_k+2\dot{v}'_k\dot{\phi}_k-\ddot{w}_k'+w_k'\dot{\phi}_k^2$$

$$+\cos\theta_I(<\ddot{\theta}_x+\dot{\theta}_y>\sin\psi_k+<\dot{\theta}_x-\ddot{\theta}_y>\cos\psi_k)+\dot{\phi}_k\dot{\theta}_z\cos\theta_I$$

$$+\ddot{\theta}_I(-\beta_s\cos\theta_I+\beta_d\sin\theta_I+1)+\dot{\theta}_I(\dot{\theta}_I<\beta_s\sin\theta_I-\beta_d\cos\theta_I>$$

$$+\dot{\phi}_k<1-\beta_d\cos\theta_I+\beta_s\sin\theta_I>)+\dot{\phi}_k\cos\theta_I\}]$$

$$-Im_{\eta\zeta}[-\cos(\theta_G+\phi_k)][\sin(\theta_G+\phi_k)\{v_k'\ddot{\phi}_k+2\dot{v}'_k\dot{\phi}_k-\ddot{w}_k'+w_k'\dot{\phi}_k^2$$

$$-\cos\theta_I(<\ddot{\theta}_x+\dot{\theta}_y>\sin\psi_k+<\dot{\theta}_x-\ddot{\theta}_y>\cos\psi_k)-\dot{\phi}_k\dot{\theta}_z\cos\theta_I$$

$$+\ddot{\theta}_I(<\beta_s\cos\theta_I-\beta_d\sin\theta_I>-1)+\dot{\theta}_I(-\dot{\theta}_I<\beta_s\sin\theta_I-\beta_d\cos\theta_I>$$

$$+\dot{\phi}_k<-1+\beta_d\cos\theta_I+\beta_s\sin\theta_I>)-\dot{\phi}_k\cos\theta_I\}+\cos(\theta_G+\phi_k)\{-\ddot{v}_k'+v_k\dot{\phi}_k^2$$

$$-2\dot{w}_k'\dot{\phi}_k-w_k'\ddot{\phi}_k-\dot{\phi}_k\cos\theta_I<\dot{\theta}_x\sin\psi_k-\dot{\theta}_y\cos\psi_k>+\ddot{\theta}_z\cos\theta_I$$

$$+\ddot{\theta}_I(1-<\beta_d\cos\theta_I+\beta_s\sin\theta_I>)$$

$$+\dot{\theta}_I(\dot{\phi}_k<\beta_s\cos\theta_I-\beta_d\sin\theta_I>+\dot{\theta}_I<\beta_d\sin\theta_I-\beta_s\cos\theta_I>-\dot{\phi}_k)+\dot{\phi}_k\sin\theta_I\}]$$

$$-m\eta_m[-\sin(\theta_G+\phi_k)][\ddot{R}_x\{\cos\psi_k-\sin\psi_k(-\beta_s\cos\theta_I+\beta_d\sin\theta_I)\}$$

$$+\dot{R}_x\{-\sin\psi_k-\cos\psi_k(-\beta_s\cos\theta_I+\beta_d\sin\theta_I)$$

$$-\dot{\theta}_I\sin\psi_k(\beta_s\sin\theta_I+\beta_d\cos\theta_I)\}+\ddot{R}_y\{\sin\psi_k+\cos\psi_k(-\beta_s\cos\theta_I+\beta_d\sin\theta_I)\}$$

$$+\dot{R}_y\{\cos\psi_k-\sin\psi_k(-\beta_s\cos\theta_I+\beta_d\sin\theta_I+\dot{\theta}_I\cos\psi_k(\beta_s\sin\theta_I+\beta_d\cos\theta_I))\}$$

$$+\ddot{R}_z(\beta_p+\beta_d\cos\theta_I+\beta_s\sin\theta_I)+\dot{R}_z\dot{\theta}_I(-\beta_d\sin\theta_I+\beta_s\cos\theta_I)$$

$$+\{a\ddot{\theta}_I(\beta_s\sin\theta_I+\beta_d\cos\theta_I)+a\dot{\theta}_I^2(\beta_s\cos\theta_I-\beta_d\sin\theta_I)+\ddot{u}_k\}$$

$$-w_k\cos\theta_I\{(\ddot{\theta}_x+\dot{\theta}_y)\sin\psi_k+(\dot{\theta}_x-\ddot{\theta}_y)\cos\psi_k\}-\dot{w}_k\cos\theta_I(\dot{\theta}_x\sin\psi_k-\dot{\theta}_y\cos\psi_k)$$

$$-\ddot{\theta}_z\cos\theta_I(a\cos\theta_I+v_k)-\dot{\theta}_z\dot{v}_k-\ddot{\theta}_I\{(-w_k+e_1+e_2)(\beta_s\cos\theta_I-\beta_d\sin\theta_I)$$

$$+(v_k+a\cos\theta_I)+(e_1+e_2-v_k-a\cos\theta_I)(\beta_d\cos\theta_I+\beta_s\sin\theta_I)+w_k\}$$

$$-\dot{\theta}_I\{-\dot{w}_k(\beta_s\cos\theta_I-\beta_d\sin\theta_I)-\dot{\theta}_I(-w_k+e_1+e_2)(\beta_s\sin\theta_I-\beta_d\cos\theta_I)$$

$$+\dot{v}_k(1-\beta_d\cos\theta_I-\beta_s\sin\theta_I)+\dot{\theta}_I(e_1+e_2-a\cos\theta_I-v_k)$$

$$(-\beta_d\sin\theta_I+\beta_s\cos\theta_I)+\dot{w}_k\}$$

$$-\dot{\theta}_I\cos\theta_I(e_1+e_2)(\beta_s\sin\theta_I+\beta_d\cos\theta_I)+\dot{v}_k$$

$$-\{(e_1+e_2)(-\beta_d\sin\theta_I+\beta_s\cos\theta_I)+\dot{w}_k\}\dot{\theta}_I\sin\theta_I]$$

$$-m\zeta_m[-\cos(\theta_G+\phi_k)][\ddot{R}_x\{\cos\psi_k-\sin\psi_k(-\beta_s\cos\theta_I+\beta_d\sin\theta_I)\}$$

$$+\dot{R}_x\{-\sin\psi_k-\cos\psi_k(-\beta_s\cos\theta_I+\beta_d\sin\theta_I)$$

$$-\dot{\theta}_I\sin\psi_k(\beta_s\sin\theta_I+\beta_d\cos\theta_I)\}+\ddot{R}_y\{\sin\psi_k+\cos\psi_k(-\beta_s\cos\theta_I+\beta_d\sin\theta_I)\}$$

$$+\dot{R}_y\{\cos\psi_k-\sin\psi_k(-\beta_s\cos\theta_I+\beta_d\sin\theta_I+\dot{\theta}_I\cos\psi_k(\beta_s\sin\theta_I+\beta_d\cos\theta_I))\}$$

$$+\ddot{R}_z(\beta_p+\beta_d\cos\theta_I+\beta_s\sin\theta_I)+\dot{R}_z\dot{\theta}_I(-\beta_d\sin\theta_I+\beta_s\cos\theta_I)$$

$$+\{a\ddot{\theta}_I(\beta_s \sin\theta_I + \beta_d \cos\theta_I) + a\dot{\theta}_I^2(\beta_s \cos\theta_I - \beta_d \sin\theta_I) + \ddot{u}_k\}$$
$$-w_k \cos\theta_I\{(\ddot{\theta}_x + \dot{\theta}_y)\sin\psi_k + (\dot{\theta}_x - \ddot{\theta}_y)\cos\psi_k\} - \dot{w}_k \cos\theta_I(\dot{\theta}_x \sin\psi_k - \dot{\theta}_y \cos\psi_k)$$
$$-\ddot{\theta}_z \cos\theta_I(a\cos\theta_I + v_k) - \dot{\theta}_z \dot{v}_k - \ddot{\theta}_I\{(-w_k + e_1 + e_2)(\beta_s \cos\theta_I - \beta_d \sin\theta_I)$$
$$+(v_k + a\cos\theta_I) + (e_1 + e_2 - v_k - a\cos\theta_I)(\beta_d \cos\theta_I + \beta_s \sin\theta_I) + w_k\}$$
$$-\dot{\theta}_I\{-\dot{w}_k(\beta_s \cos\theta_I - \beta_d \sin\theta_I) - \dot{\theta}_I(-w_k + e_1 + e_2)(\beta_s \sin\theta_I - \beta_d \cos\theta_I)$$
$$+\dot{v}_k(1 - \beta_d \cos\theta_I - \beta_s \sin\theta_I) + \dot{\theta}_I(e_1 + e_2 - a\cos\theta_I - v_k)(-\beta_d \sin\theta_I + \beta_s \cos\theta_I) + \dot{w}_k\}$$
$$-\dot{\theta}_I \cos\theta_I(e_1 + e_2)(\beta_s \sin\theta_I + \beta_d \cos\theta_I) + \dot{v}_k$$
$$-\{(e_1 + e_2)(-\beta_d \sin\theta_I + \beta_s \cos\theta_I) + \dot{w}_k\}\dot{\theta}_I \sin\theta_I]$$

The section integrals are defined as:

$$m = \int\int_A \rho \, d\eta \, d\zeta \tag{A.1}$$

$$m\eta_m = \int\int_A \rho\eta \, d\eta \, d\zeta \tag{A.2}$$

$$m\zeta_m = \int\int_A \rho\zeta \, d\eta \, d\zeta \tag{A.3}$$

$$Im_{\eta\eta} = \int\int_A \rho\zeta^2 \, d\eta \, d\zeta \tag{A.4}$$

$$Im_{\zeta\zeta} = \int\int_A \rho\eta^2 \, d\eta \, d\zeta \tag{A.5}$$

$$Im_{\eta\zeta} = \int\int_A \rho\eta\zeta \, d\eta \, d\zeta \tag{A.6}$$

Where m is the mass per unit length of the blade; $m\eta_m$ and $m\zeta_m$ are the first moment of the mass of the cross-section about the elstic axis; $Im_{\eta\eta}$, $Im_{\zeta\zeta}$ and $Im_{\eta\zeta}$ are the mass moments of inertia per unit length of the beam about the elastic axis.

Appendix B

MODIFICATION OF THE TERM $\left[V_{31}^{L}\right]$ ASSOCIATED WITH KINETIC ENERGY

Expanding $\sin(\theta_G + \phi_k)$ and $\cos(\theta_G + \phi_k)$, and assuming that ϕ_k is small-

$$\sin(\theta_G + \phi_k) \approx \sin\theta_G + \phi_k \cos\theta_G$$

$$\cos(\theta_G + \phi_k) \approx \cos\theta_G - \phi_k \sin\theta_G$$

Substituting the above approximation in $\left[V_{31}^{L}\right]$, and rewriting as

$$\left[V_{31}^{L}\right] \approx \left[V_{31}^{L}\#\right] + \left[K_{33}^{cf}\#\right]\{\Phi\}$$

where

$$\begin{aligned}
\left[K_{33}^{cf}\#\right] &= \int_0^{l_r} \Big[\{(Im_{\zeta\zeta} - Im_{\eta\eta})(\cos^2\theta_G - \sin^2\theta_G)\cos^2\theta_I \\
&\quad +4Im_{\eta\zeta}\sin\theta_G\cos\theta_G\cos^2\theta_I + (Im_{\zeta\zeta}\cos\theta_G - Im_{\eta\eta}\sin\theta_G)\sin\theta_I\}\ \{\Phi_q\}\ \{\Phi_q\}^T\Big]\, dx
\end{aligned}$$

$$\begin{aligned}
\left[V_{31}^{L}\#\right] &= \int_0^{l_r} \Big[\{(Im_{\zeta\zeta} - Im_{\eta\eta})\sin\theta_G\cos\theta_G\cos^2\theta_I - Im_{\eta\zeta}(\sin^2\theta_G - \cos^2\theta_G)\cos^2\theta_I \\
&\quad +(Im_{\zeta\zeta}\sin\theta_G + Im_{\eta\eta}\cos\theta_G)\sin\theta_I\}\ \{\Phi_q\}\Big]\, dx
\end{aligned}$$

The submatrix $\left[K_{33}^{cf}\#\right]$ has to be added to the matrix $\left[K_{33}^{cf}\right]$ of linear stiffness matrix, given in Sec. 6.2.3.